THE ELEMENTARY PART

OF A TREATISE ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES

BEING PART I. OF A TREATISE ON THE WHOLE SUBJECT

With numerous Examples

BY

EDWARD JOHN ROUTH, Sc.D., LL.D., F.R.S., &c.

HON. FELLOW OF PETERHOUSE, CAMBRIDGE;

FELLOW OF THE SENATE OF THE UNIVERSITY OF LONDON.

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PREFACE.

THE opportunity of a new edition has enabled the author to make numerous additions to both the volumes of this treatise. To make room for these some less important matter has been omitted. Many of these additions have already appeared in the German translation of this work and this is particularly the case with the additions made to the second volume. In the seven or eight years which have elapsed since the translation was published the progress of the science has not been slow. Much new matter therefore has been introduced into both the volumes and this has been arranged either as new theorems or as examples according to their importance.

The dynamical principles of the subject are given in this volume together with the more elementary applications, while the more difficult theories and problems appear in the second. Sometimes one case of a problem supplies an example sufficiently elementary to appear in this volume while the general theory is given in the next. For example, the small oscillations of a vertical top and the motion of a sphere on a rough plane are partly discussed here, but they are more fully treated of in the second volume. In order that the plan of the book may be understood, a short summary of the next volume has been added to the table of contents.

Each chapter has been made as far as possible complete in itself. This arrangement is convenient for those who are already acquainted with dynamics, as it enables them to direct their attention to those parts in which they may feel most interested. It also enables the student to select his own order of reading.

The student who is just beginning dynamics may not wish to be delayed by a chapter of preliminary analysis before he enters on the real subject of the book. He may therefore begin with D'Alembert's Principle and read only those parts of Chapter I. to which reference is made. Others may wish to pass on as soon as possible to the principles of Angular Momentum and Vis Viva. Though a different order may be found advisable for some readers, I have ventured to indicate a list of Articles to which those who are beginning dynamics should first turn their attention.

As in the previous editions a chapter has been devoted to the discussion of Motion in Two Dimensions. This course has been adopted because it seemed expedient to separate the difficulties of dynamics from those of solid geometry.

Throughout each chapter there will be found numerous examples, many being very easy, while others are intended for the more advanced student. In order to obtain as great a variety of problems as possible, a collection has been added at the end of each chapter, taken from the Examination Papers which have been set in the University and in the Colleges. As these problems have been constructed by many different examiners, it is hoped that this selection will enable the student to acquire facility in solving all kinds of dynamical problems.

There are many useful instruments and important experimental researches whose theories require only a knowledge of dynamics and which can be easily understood without any long or intricate description. It will be seen that many of these have been selected as useful examples.

Historical sketches have been attempted whenever they could be briefly given. Such notices, if not carried too far, add greatly to the interest of the subject. It is chiefly with the memoirs written since the early part of the last century that we are here concerned, and the number of these is so great that anything more than a slight notice of some of them is impossible.

A useful theorem is many times discovered and probably each time with variations. It is thus often difficult to ascertain who is the real author. It has therefore been found necessary to correct some of the references given in the former editions and to add references where there were none before.

The use of dots and accents for differential coefficients with regard to the time has been continued whenever a short notation was desirable. One objection to this notation is that the meaning of the symbol may be greatly changed by a slight error in the number of the dots or accents. As this might increase the difficulties of the subject to a beginner, the use of dots in the earlier chapters has been restricted chiefly to the working of examples, and care has been taken that the results should be clearly stated.

EDWARD J. ROUTH.

PETERHOUSE,
August 1905.

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NOTES.

The following subjects will be treated of in the second volume.

Theory of moving axes, Clairaut's theorem, motion relative to the earth, and gyroscopes.

Theory of small oscillations with several degrees of freedom both about a position of equilibrium and about a state of steady motion.

Motion of a body about a fixed point under no forces.

Motion of a body under any forces, top, sphere, solid of revolution, any solid.

Linear equations, conditions (1) for the absence of powers of the time, and (2) for stability.

Theory of free and forced oscillations.

Methods of Isolation and of Multipliers.

Applications of the calculus of finite differences, chain and network of particles.

Applications of the calculus of variations, Hamilton, Jacobi, Lagrange, &c.

Precession and Nutation.

Motion of the Moon about its centre.

Motion of a string or chain, (1) loose, (2) tight.

Impact and Vibrations of elastic rods.

Motion of a membrane, (1) homogeneous, (2) heterogeneous.

Conjugate functions applied to vortex motion.

The student, to whom the subject is entirely new, is advised to read *first* the following articles: Chap. I. 1—25, 33—36, 47—52. Chap. II. 66—87. Chap. III. 88—93, 98—104, 110, 112—118. Chap. IV. 130—164, 168—174, 179—186, 199. Chap. V. 214—245, 248—256, 261—269. Chap. VI. 282—285, 287—295, 299—304, 306—309. Chap. VII. 332—373. Chap. VIII. 395—409. Chap. IX. 432—463, 467—476. Chap. X. 483, 488—499.

ERRATUM IN VOL. II.

Page 458, line 23. *For* "To these oscillations we add the complementary function" *read* "with these oscillations we compare those of the unloaded membrane."

CHAPTER I

MOMENTS OF INERTIA

1. In the subsequent pages of this work it will be found that certain integrals continually recur. It is therefore convenient to collect these into a preliminary chapter for reference. Though their bearing on dynamics may not be obvious beforehand, yet the student may be assured that it is as useful to be able to write down moments of inertia with facility as it is to be able to quote the centres of gravity of the elementary bodies.

In addition however to these necessary propositions there are many others which are useful as giving a more complete view of the arrangement of the axes of inertia in a body. These also have been included in this chapter though they are not of the same importance as the former.

2. All the integrals used in dynamics as well as those used in statics and some other branches of mixed mathematics are included in the one form

$$\iiint x^\alpha y^\beta z^\gamma \, dx \, dy \, dz,$$

where (α, β, γ) have particular values. In statics two of these three exponents are usually zero, and the third is either unity or zero, according as we wish to find the numerator or denominator of a coordinate of the centre of gravity. In dynamics of the three exponents one is zero, and the sum of the other two is usually equal to 2. The integral in all its generality has not yet been fully discussed, probably because only certain cases have any real utility. In the case in which the body considered is a homogeneous ellipsoid the value of the general integral has been found in gamma functions by Lejeune Dirichlet in Vol. IV. of *Liouville's Journal*. His results were afterwards extended by Liouville in the same volume to the case of a heterogeneous ellipsoid in which the strata of uniform density are similar ellipsoids.

In this treatise, it is intended chiefly to restrict ourselves to the consideration of moments and products of inertia, as being the only cases of the integral which are useful in dynamics.

efinitions. If the mass of every particle of a material ..s multiplied by the square of its distance from a straight .ie sum of the products so formed is called the *moment of ıa* of the system about that line.

The If M be the mass of a system and k be such a quantity that $.k^2$ is its moment of inertia about a given straight line, then k is called the *radius of gyration* of the system about that line.

The term "moment of inertia" was introduced by Euler, and has now come into general use wherever Rigid Dynamics is studied. It will be convenient for us to use the following additional terms.

If the mass of every particle of a material system is multiplied by the square of its distance from a given plane or from a given point, the sum of the products so formed is called the moment of inertia of the system with reference to that plane or that point.

If two straight lines Ox, Oy be taken as axes, and if the mass of every particle of the system be multiplied by its *two* coordinates x, y, the sum of the products so formed is called the *product of inertia* of the system about those two axes.

This might, perhaps more conveniently, be called the product of inertia of the system with reference to the two coordinate planes xz, yz.

The term *moment of inertia with regard to a plane* seems to have been first used by M. Binet in the *Journal Polytechnique*, 1813.

4. Let a body be referred to any rectangular axes Ox, Oy, Oz meeting in a point O, and let x, y, z be the coordinates of any particle m, then according to these definitions the moments of inertia about the axes of x, y, z respectively will be

$$A = \Sigma m (y^2 + z^2), \quad B = \Sigma m (z^2 + x^2), \quad C = \Sigma m (x^2 + y^2).$$

The moments of inertia with regard to the planes yz, zx, xy, respectively, will be

$$A' = \Sigma mx^2, \quad B' = \Sigma my^2, \quad C' = \Sigma mz^2.$$

The products of inertia with regard to the axes yz, zx, xy will be

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad F = \Sigma mxy.$$

Lastly, the moment of inertia with regard to the origin will be

$$H = \Sigma m (x^2 + y^2 + z^2) = \Sigma mr^2,$$

where r is the distance of the particle m from the origin.

5. **Elementary Propositions.** The following propositions may be established without difficulty, and will serve as illustrations of the preceding definitions.

(1) The three moments of inertia A, B, C about three rectangular axes are such that the sum of any two of them is greater than the third. For $A+B-C=2\Sigma mz'^2$ and is positive.

(2) The sum of the moments of inertia about any three rectangular axes meeting at a given point is always the same; and is equal to twice the moment of inertia with respect to that point. For $A+B+C=2\Sigma m(x^2+y^2+z^2)=2\Sigma mr^2$, and is therefore independent of the directions of the axes.

(3) The sum of the moments of inertia of a system with reference to any plane through a given point and its normal at that point is constant and equal to the moment of inertia of the system with reference to that point. Take the given point as origin and the plane as the plane of xy, then $C'+C=\Sigma mr^2$, which is independent of the directions of the axes.

Hence we infer that

$$A'=\tfrac{1}{2}(B+C-A),\quad B'=\tfrac{1}{2}(C+A-B),\quad \text{and}\quad C'=\tfrac{1}{2}(A+B-C).$$

(4) Any product of inertia as D cannot numerically be so great as $\frac{1}{2}A$.

(5) If A, B, F are the moments and product of inertia of a lamina about two rectangular axes in its plane, then AB is greater than F^2. If t be any quantity we have $At^2+2Ft+B=\Sigma m(yt+x)^2=$ a positive quantity. Hence the roots of the quadratic $At^2+2Ft+B=0$ are imaginary, and therefore $AB>F^2$.

(6) Prove that for any body

$$(A+B-C)(B+C-A)>4E^2,$$
$$(A+B-C)(B+C-A)(C+A-B)>8DEF.$$

(7) The moment of inertia of the surface of a sphere of radius a and mass M about any diameter is $M\frac{2}{3}a^2$. Since every element is equally distant from the centre its moment of inertia about the centre is Ma^2. Hence by (2) the result follows.

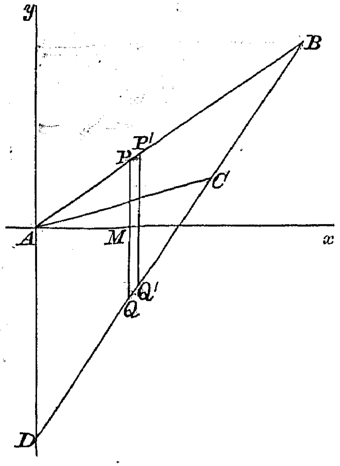
(8) The moment of inertia of the surface of a hemisphere of radius a and mass M about every diameter is $M\frac{2}{3}a^2$. This follows immediately from (7) by completing the sphere, writing $2M$ for M and halving the result.

6. It is clear that the process of finding moments and products of inertia is merely that of integration. We may illustrate this by the following example.

To find the moment of inertia of a uniform triangular plate about an axis in its plane passing through one angular point.

Let ABC be the triangle, Ay the axis about which the moment is required. Draw Ax perpendicular to Ay and produce BC to meet Ay in D. The given triangle ABC may be regarded as the

difference of the triangles ABD, ACD. Let us then first find the moment of inertia of ABD. Let $PQP'Q'$ be an elementary area whose sides PQ, $P'Q'$ are parallel to the base AD, and let PQ cut Ax in M. Let β be the distance of the angular point B from the axis Ay, $AM = x$ and $AD = l$.



Then the elementary area $PQP'Q'$ is clearly $l\dfrac{\beta - x}{\beta}dx$, and its moment of inertia about Ay is $\mu l\dfrac{\beta - x}{\beta}dx \,.\, x^2$, where μ is the mass per unit of area. Hence the moment of inertia of the triangle ABD

$$= \mu\int_0^\beta l\left(1 - \frac{x}{\beta}\right)x^2dx = \tfrac{1}{12}\mu l\beta^3.$$

Similarly if γ be the distance of the angular point C from the axis Ay, the moment of inertia of the triangle ACD is $\frac{1}{12}\mu l\gamma^3$. Hence the moment of inertia of the given triangle ABC is $\frac{1}{12}\mu l(\beta^3 - \gamma^3)$. Now $\frac{1}{2}l\beta$ and $\frac{1}{2}l\gamma$ are the areas of the triangles ABD, ACD. Hence if M be the mass of the triangle ABC, the moment of inertia of the triangle about the axis Ay is

$$\tfrac{1}{6}M(\beta^2 + \beta\gamma + \gamma^2).$$

Ex. If each element of the mass of the triangle be multiplied by the nth power of its distance from the straight line through the angle A, then it may be proved in the same way that the sum of the products is $\dfrac{2M}{(n+1)(n+2)}\dfrac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma}$.

7. *When the body is a lamina the moment of inertia about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two rectangular axes in its plane drawn from the point where the former axis meets the plane.*

For let the axis of z be taken normal to the plane, then, if A, B, C are the moments of inertia about the axes, we have

$$A = \Sigma my^2, \quad B = \Sigma mx^2, \quad C = \Sigma m(x^2 + y^2),$$

and therefore $$C = A + B.$$

We may apply this theorem to the case of the triangle. Let β', γ' be the distances of the points B, C from the axis Ax. Then the moment of inertia of the triangle about a normal to the plane of the triangle through the point A is

$$\tfrac{1}{6}M(\beta^2 + \beta\gamma + \gamma^2 + \beta'^2 + \beta'\gamma' + \gamma'^2).$$

Ex. Prove that the moment of inertia of the perimeter of a circle of radius a and mass M about any diameter is $\frac{1}{2}Ma^2$.

Since every element is equally distant from the axis of the circle, the moment of inertia about that axis is $C = Ma^2$. Since $A = B$, the result follows at once.

8. **Reference Table.** The following moments of inertia occur so frequently that they have been collected together for reference. The reader is advised to commit to memory the following table:

The moment of inertia of

(1) A rectangle whose sides are $2a$ and $2b$

about an axis through its centre in its plane perpendicular to the side $2a$ $\Big\} = \text{mass}\,\dfrac{a^2}{3}$,

about an axis through its centre perpendicular to its plane $\Big\} = \text{mass}\,\dfrac{a^2+b^2}{3}$.

(2) An ellipse semi-axes a and b

$$\text{about the major axis } a = \text{mass}\,\frac{b^2}{4},$$

$$\text{about the minor axis } b = \text{mass}\,\frac{a^2}{4},$$

about an axis perpendicular to its plane through the centre $\Big\} = \text{mass}\,\dfrac{a^2+b^2}{4}$.

In the particular case of a circle of radius a, the moment of inertia about a diameter $= \text{mass}\,\dfrac{a^2}{4}$, and that about a perpendicular to its plane through the centre $= \text{mass}\,\dfrac{a^2}{2}$.

(3) An ellipsoid semi-axes a, b, c

$$\text{about the axis } a = \text{mass}\,\frac{b^2+c^2}{5}.$$

In the particular case of a sphere of radius a the moment of inertia about a diameter $= \text{mass}\,\dfrac{2}{5}a^2$.

(4) A right solid whose sides are $2a$, $2b$, $2c$

about an axis through its centre perpendicular to the plane containing the sides b and c $\Big\} = \text{mass}\,\dfrac{b^2+c^2}{3}$.

These results may be all included in one rule, which the author has long used as an assistance to the memory.

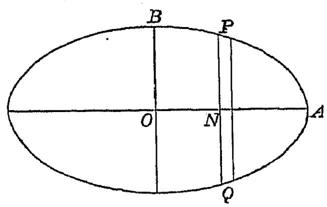
$$\left.\begin{array}{r}\text{Moment of inertia}\\ \text{about an axis}\\ \text{of symmetry}\end{array}\right\} = \text{mass}\,\frac{\text{(sum of squares of perpendicular semi-axes)}}{3,\ 4 \text{ or } 5}.$$

The denominator is to be 3, 4 or 5, according as the body is rectangular, elliptical or ellipsoidal.

Thus, if we require the moment of inertia of a circle of radius a about a diameter, we notice that the perpendicular semi-axis in its plane is the radius a, and that the semi-axis perpendicular to its plane is zero, the moment of inertia required is therefore $M\frac{a^2}{4}$, if M be the mass. If we require the moment about a perpendicular to its plane through the centre, we notice that the perpendicular semi-axes are each equal to a and the moment required is therefore $M\frac{a^2+a^2}{4}=M\frac{a^2}{2}$.

9. As the process for determining these moments of inertia is very nearly the same for all these cases, it will be sufficient to consider only two instances.

To determine the moment of inertia of an ellipse about the minor axis.



Let the equation of the ellipse be $y=\frac{b}{a}\sqrt{a^2-x^2}$. Take any elementary area PQ parallel to the axis of y, then clearly the moment of inertia is

$$4\mu\int_0^a x^2 y\,dx=4\mu\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx,$$

where μ is the mass of a unit of area.

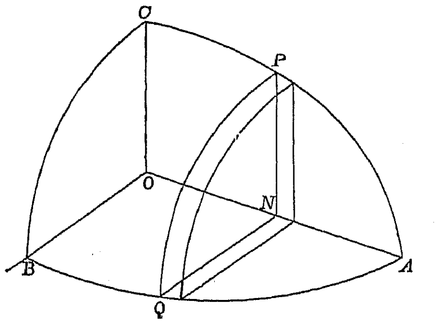
To integrate this, put $x=a\sin\phi$, and the integral becomes

$$a^4\int_0^{\frac{\pi}{2}}\cos^2\phi\sin^2\phi\,d\phi=a^4\int_0^{\frac{\pi}{2}}\frac{1-\cos4\phi}{8}\,d\phi=\frac{\pi a^4}{16};$$

$$\therefore\ \text{the moment of inertia}=\mu\pi ab\frac{a^2}{4}=\text{mass}\,\frac{a^2}{4}.$$

In the same way we may show that the product of inertia of an elliptic quadrant about its axis $=\text{mass}\,\frac{ab}{2\pi}$.

To determine the moment of inertia of an ellipsoid about a principal diameter.



Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Take any elementary area PNQ parallel to the plane of yz. Its area is evidently $\pi PN\,.\,QN$. Now PN is the value of z when $y=0$, and QN the value of y when $z=0$, as obtained from the equation of the ellipsoid;

$$\therefore\ PN=\frac{c}{a}\sqrt{a^2-x^2},$$

$$QN=\frac{b}{a}\sqrt{a^2-x^2};$$

$$\therefore\ \text{the area of the element}=\frac{\pi bc}{a^2}(a^2-x^2).$$

Let μ be the mass of the unit of volume, then the whole moment of inertia

$$=\mu\int_{-a}^{a}\frac{\pi bc}{a^2}(a^2-x^2)\frac{PN^2+QN^2}{4}\,dx=\mu\frac{\pi}{4}\frac{bc}{a^2}\int_{-a}^{a}(a^2-x^2)\frac{b^2+c^2}{a^2}(a^2-x^2)\,dx$$

$$=\mu\frac{4}{3}\pi abc\,\frac{b^2+c^2}{5}=\text{mass}\,\frac{b^2+c^2}{5}.$$

In the same way we may show that the product of inertia of the octant of an ellipsoid about the axis of $(x, y)=\text{mass}\,\frac{2ab}{5\pi}$.

Ex. 1. The moment of inertia of an arc of a circle whose radius is a and which subtends an angle 2α at the centre about an axis

(a) through its centre perpendicular to its plane $=Ma^2$,

(b) through its middle point perpendicular to its plane $=2M\left(1-\frac{\sin\alpha}{\alpha}\right)a^2$,

(c) about the diameter which bisects the arc $=M\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\frac{a^2}{2}$.

Ex. 2. The moment of inertia of the part of the area of a parabola cut off by any ordinate at a distance x from the vertex is $\frac{3}{7}Mx^2$ about the tangent at the vertex, and $\frac{1}{5}My^2$ about the principal diameter, where y is the ordinate corresponding to x.

Ex. 3. The moment of inertia of the area of the lemniscate $r^2=a^2\cos 2\theta$ about a line through the origin in its plane and perpendicular to its axis is $Ma^2(3\pi+8)/48$.

Ex. 4. A lamina is bounded by four rectangular hyperbolas, two of them have the axes of coordinates for asymptotes, and the other two have the axes for principal diameters. Prove that the sum of the moments of inertia of the lamina about the coordinate axes is $\frac{1}{4}(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)$, where $\alpha, \alpha'; \beta, \beta'$ are the semi-major axes of the hyperbolas.

Take the equations $xy=u$, $x^2-y^2=v$, then the two moments of inertia are $B=\iint x^2 J\,du\,dv$ and $A=\iint y^2 J\,du\,dv$, where $1/J$ is the Jacobian of (u, v) with regard to (x, y). This gives at once $A+B=\frac{1}{2}\iint du\,dv$, where the limits are clearly $u=\frac{1}{2}\alpha^2$ to $u=\frac{1}{2}\alpha'^2$, $v=\beta^2$ to $v=\beta'^2$.

Ex. 5. A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two similar hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the coordinate axes. Prove that the product of inertia about the coordinate axes is $\frac{(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)}{4(m^2+n^2)}$, where $\alpha, \alpha'; \beta, \beta'$ are the semi-major axes of the curves.

Ex. 6. If $d\sigma$ is an element of the surface of a sphere referred to any rectangular axes meeting at the centre, prove that $\int x^{2n}\,d\sigma=\frac{4\pi}{2n+1}r^{2n+2}$, where r is the radius of the sphere and n is integral.

Ex. 7. Taking the same axes as in the last example, prove that

$$\int x^{2f}y^{2g}z^{2h}\,d\sigma=\frac{4\pi}{2n+1}r^{2n+2}\frac{L(f)\,L(g)\,L(h)}{L(n)},$$

where $n=f+g+h$ and $L(f)$ stands for the quotient of the product of all the natural numbers up to $2f$ by the product of the same numbers up to f, both included.

To prove this, we notice that by the last example we have

$$\int(\lambda x+\mu y+\nu z)^{2n}\,d\sigma=(\lambda^2+\mu^2+\nu^2)^n\,\frac{4\pi r^{2n+2}}{2n+1}.$$

Expand both sides and equate the coefficients of $\lambda^{2f}\mu^{2g}\nu^{2h}$.

If we multiply the result by Ddr we have the value of the integral for any homogeneous shell of density D and thickness dr. Regarding D as a function of r, and integrating with regard to r, we can find the value of the integral for any heterogeneous sphere in which the strata of equal density are concentric spheres.

Ex. 8. If $d\sigma$ is an element of the surface of an ellipsoid referred to its principal diameters, and if p is the perpendicular from the centre on the tangent plane, prove

$$\int x^{2f}y^{2g}z^{2h}p\,d\sigma = \frac{4\pi}{2n+1}\,\frac{L(f)\,L(g)\,L(h)}{L(n)}\,a^{2f+1}\,b^{2g+1}\,c^{2h+1},$$

where a, b, c are the semi-axes and the rest of the notation is the same as before.

This result follows at once from the corresponding one for a spherical shell by the *method of projections*. The corresponding integral when the indices of x, y, z are any quantities and the integration extends over an octant of the surface is given by Dirichlet's theorem in gamma functions.

Ex. 9. Show that the volume V, the surface S, and the moment of inertia I with regard to the plane perpendicular to the coordinate x_1, of the sphere in space of n dimensions, whose equation is $x_1^2+x_2^2+\ldots+x_n^2=r^2$, are given by

$$V=r^n\,(\Gamma\tfrac{1}{2})^n/\Gamma\,(\tfrac{1}{2}n+1),\qquad S=\frac{n}{r}V,\qquad I=V\frac{r^2}{n+2}.$$

These results follow easily from Dirichlet's theorem. See also Art. 5 (2).

10. **Method of Differentiation.** Many moments of inertia may be deduced from those given in Art. 8 by the method of differentiation. Thus the moment of inertia of a solid ellipsoid of uniform density ρ about the axis of a is known to be $\frac{4}{3}\pi abc\rho\,\frac{b^2+c^2}{5}$. Let the ellipsoid increase indefinitely little in size, then the moment of inertia of the enclosed shell is $d\left\{\frac{4}{3}\pi abc\rho\,\frac{b^2+c^2}{5}\right\}$.

This differentiation can be effected *as soon as the law according to which the ellipsoid alters is given.* Suppose the bounding ellipsoids to be similar, and let the ratio of the axes in each be given by $b=pa$, $c=qa$. Then

$$\text{moment of inertia of solid ellipsoid} = \tfrac{4}{3}\pi\rho pq\,\frac{p^2+q^2}{5}a^5;$$

$$\therefore\ \text{moment of inertia of shell} = \tfrac{4}{3}\pi\rho pq\,(p^2+q^2)\,a^4da.$$

In the same way the mass of solid ellipsoid $=\frac{4}{3}\pi\rho pqa^3$;

$$\therefore\ \text{mass of shell} = 4\pi\rho pqa^2da.$$

Hence the moment of inertia of an indefinitely thin ellipsoidal shell of mass M bounded by similar ellipsoids is $\frac{1}{3}M\,(b^2+c^2)$.

By reference to Art. 8, it will be seen that this is the same as the moment of inertia of the circumscribing right solid of equal mass. *These two bodies therefore have equal moments of inertia about their axes of symmetry at the centre of gravity.*

11. The moments of inertia of a heterogeneous body whose boundary is a surface of uniform density may sometimes be found by the method of differentiation. Suppose the moment of inertia of a homogeneous body of density D, bounded by any surface of uniform density, to be known. Let this when expressed in terms of some parameter a be $\phi(a) D$. Then the moment of inertia of a stratum of density D will be $\phi'(a)\, Dda$. Replacing D by the variable density ρ, the moment of inertia required will be $\int \rho\phi'(a)\, da$.

Ex. 1. Show that the moment of inertia of a heterogeneous ellipsoid about the major axis, the strata of uniform density being similar concentric ellipsoids, and the density along the major axis varying as the distance from the centre, is $\frac{2}{9} M (b^2+c^2)$.

Ex. 2. The moment of inertia of a heterogeneous ellipse about the minor axis, the strata of uniform density being confocal ellipses and the density along the minor axis varying as the distance from the centre, is $\dfrac{3M}{20}\dfrac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}$.

12. **Other methods of finding moments of inertia.** The moments of inertia given in the table in Art. 8 are only a few of those in continual use. The moments of inertia of an ellipse, for example, about its principal axes are there given, but we shall also frequently want its moments of inertia about other axes. It is of course possible to find these in each separate case by integration. But this is a tedious process, and it may be often avoided by the use of the two following propositions.

The moments of inertia of a body about certain axes through its centre of gravity, which we may take as axes of reference, are regarded as given in the table. In order to find the moment of inertia of *that body* about any other axis we shall investigate:

(1) A method of comparing the required moment of inertia with that about a parallel axis through the centre of gravity. This is the theorem of parallel axes.

(2) A method of determining the moment of inertia about this parallel axis in terms of the given moments of inertia about the axes of reference. This is the theorem of the six constants of a body.

13. **Theorem of Parallel Axes.** *Given the moments and products of inertia about all axes through the centre of gravity of a body, to deduce the moments and products about all parallel axes.*

The moment of inertia of a system of bodies about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus the moment of inertia of the whole mass collected at the centre of gravity about the original axis.

The product of inertia about any two axes is equal to the product of inertia about two parallel axes through the centre of

gravity plus the product of inertia of the whole mass collected at the centre of gravity about the original axes.

Firstly, take the axis about which the moment of inertia is required as the axis of z. Let m be the mass of any particle of the body, which generally will be any small element. Let x, y, z be the coordinates of m, $\bar{x}, \bar{y}, \bar{z}$ those of the centre of gravity G of the whole system of bodies, x', y', z' those of m referred to a system of parallel axes through the centre of gravity.

Then since $\dfrac{\Sigma mx'}{\Sigma m}, \dfrac{\Sigma my'}{\Sigma m}, \dfrac{\Sigma mz'}{\Sigma m}$ are the coordinates of the centre of gravity of the system referred to the centre of gravity as the origin, it follows that $\Sigma mx' = 0$, $\Sigma my' = 0$, $\Sigma mz' = 0$.

The moment of inertia of the system about the axis of z is

$$\begin{aligned} &= \Sigma m (x^2 + y^2), \\ &= \Sigma m \{(\bar{x} + x')^2 + (\bar{y} + y')^2\}, \\ &= \Sigma m (\bar{x}^2 + \bar{y}^2) + \Sigma m (x'^2 + y'^2) + 2\bar{x} \,.\, \Sigma mx' + 2\bar{y} \,.\, \Sigma my'. \end{aligned}$$

Now $\Sigma m (\bar{x}^2 + \bar{y}^2)$ is the moment of inertia of a mass Σm collected at the centre of gravity, and $\Sigma m (x'^2 + y'^2)$ is the moment of inertia of the system about an axis through G, also $\Sigma mx' = 0$, $\Sigma my' = 0$; whence the proposition is proved.

It follows from this theorem, that, *of all axes parallel to a given straight line that one has the least moment of inertia which passes through the centre of gravity.*

Secondly, take the axes of x, y as the axes about which the product of inertia is required. The product required is

$$\begin{aligned} &= \Sigma m\, xy = \Sigma m (\bar{x} + x')(\bar{y} + y'), \\ &= \bar{x}\bar{y} \,.\, \Sigma m + \Sigma mx'y' + \bar{x}\Sigma my' + \bar{y}\Sigma mx', \\ &= \bar{x}\bar{y}\Sigma m + \Sigma mx'y'. \end{aligned}$$

Now $\bar{x}\bar{y} \,.\, \Sigma m$ is the product of inertia of a mass Σm collected at G and $\Sigma mx'y'$ is the product of the whole system about axes through G; whence the proposition is proved.

Let there be two parallel axes A and B at distances a and b from the centre of gravity of the body. Then, if M be the mass of the material system,

$$\left.\begin{matrix}\text{moment of inertia} \\ \text{about } A\end{matrix}\right\} - Ma^2 = \left\{\begin{matrix}\text{moment of inertia} \\ \text{about } B\end{matrix}\right. - Mb^2.$$

Hence when the moment of inertia of a body about one axis is known, that about any other parallel axis may be found. It is obvious that a similar proposition holds with regard to the products of inertia.

14. *The preceding proposition may be generalized as follows.* Let any system be in motion, and let x, y, z be the coordinates at the time t of any particle of mass m. Let also $\dot{x}, \dot{y}, \dot{z}$; $\ddot{x}, \ddot{y}, \ddot{z}$ be the resolved velocities and accelerations of the same particle, where the dots represent as usual differentiations with regard to the time. Suppose

$$V = \Sigma m\phi\,(x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}, z, \dot{z}, \ddot{z})$$

to be a given function depending on the structure and motion of the system, the summation extending throughout the system. Also let ϕ be an algebraic function of the first or second order. Thus ϕ may consist of such terms as

$$ax^2 + bx\dot{y} + c\dot{z}^2 + eyz + fx + \ldots\ldots$$

where a, b, c, &c. are some constants. Then the following general principle will hold.

The value of V for any system of coordinates is equal to the value of V obtained for a parallel system of coordinates with the centre of gravity for origin plus the value of V for the whole mass collected at the centre of gravity with reference to the first system of coordinates.

For let $\bar{x}, \bar{y}, \bar{z}$ be the coordinates of the centre of gravity, and let $x = \bar{x} + x'$, &c., $\therefore \dot{x} = \dot{\bar{x}} + \dot{x}'$, &c.

Now since ϕ is an algebraic function of the second order of $x, \dot{x}, \ddot{x}$; y, &c. it is evident that on making the above substitution and expanding, the process of squaring &c. will lead to three sets of terms, those containing only $\bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}$, &c., those containing the products of $\bar{x}, x'$, &c., and lastly those containing only $x', \dot{x}'$, &c. The first of these will on the whole make up $\phi\,(\bar{x}, \dot{\bar{x}}$, &c.), and the last $\phi\,(x', \dot{x}'$, &c.).

Hence $$V = \Sigma m\phi\,(\bar{x}, \dot{\bar{x}}, \ldots) + \Sigma m\phi\,(x', \dot{x}' + \ldots)$$
$$+ \Sigma m\,(A\bar{x}\dot{x}' + B\dot{\bar{x}}x' + C\bar{x}\dot{y}' - \ldots),$$

where A, B, C, &c. are some constants.

Now the term $\Sigma m\,(\bar{x}\dot{x}')$ is the same as $\bar{x}\Sigma m\dot{x}'$, and this vanishes. For since $\Sigma mx' = 0$, it follows that $\Sigma m\dot{x}' = 0$. Similarly all the other terms in the second line vanish.

Hence the value of V is reduced to two terms. But the first of these is the value of V for the whole mass collected at the centre of gravity, and the second of these the value of V for the whole system referred to the centre of gravity as origin. Hence the proposition is proved.

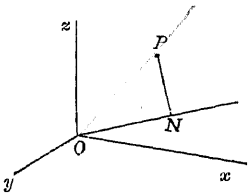
The proposition would obviously be true if $\ddot{x}, \ddot{y}, \ddot{z}$, or any higher differential coefficients were also present in the function V.

15. **Theorem of the six constants of a body.** *Given the moments and products of inertia about three straight lines at right angles meeting in a point, to deduce the moments and products of inertia about all other axes meeting in that point.*

Take these three straight lines as the axes of coordinates. Let A, B, C be the moments of inertia about the axes of x, y, z; D, E, F the products of inertia about the axes of yz, zx, xy. Let α, β, γ be the direction-cosines of any straight line through the origin, then the moment of inertia I of the body about that line will be given by the equation

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

Let P be any point of the body at which a mass m is situated, and let x, y, z be the coordinates of P. Let ON be the line whose direction-cosines are α, β, γ, draw PN perpendicular to ON.



Since ON is the projection of OP, it is clearly $= x\alpha + y\beta + z\gamma$, also

$$OP^2 = x^2 + y^2 + z^2, \text{ and } 1 = \alpha^2 + \beta^2 + \gamma^2.$$

The moment of inertia I about $ON = \Sigma mPN^2$

$$\begin{aligned}
&= \Sigma m \{x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2\} \\
&= \Sigma m \{(x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)^2\} \\
&= \Sigma m (y^2 + z^2)\alpha^2 + \Sigma m (z^2 + x^2)\beta^2 + \Sigma m (x^2 + y^2)\gamma^2 \\
&\qquad - 2\Sigma myz \,.\, \beta\gamma - 2\Sigma mzx \,.\, \gamma\alpha - 2\Sigma mxy \,.\, \alpha\beta \\
&= A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.
\end{aligned}$$

It may be shown in exactly the same manner that if A', B', C' be the moments of inertia with regard to the *planes* yz, zx, xy, that the moment of inertia with regard to the *plane* whose direction-cosines are α, β, γ is

$$I' = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta.$$

It should be remarked that this formula differs from that giving the moment of inertia about a straight line in the signs of the three last terms.

16. When three straight lines at right angles and meeting in a given point are such that if they be taken as axes of coordinates all the products Σmxy, Σmyz, Σmzx vanish, these are said to be *Principal Axes* at the given point.

The three planes which pass each through two principal axes are called the *Principal Planes* at the given point.

The moments of inertia about the principal axes at any point are called the *Principal moments of inertia* at that point.

The fundamental formula in Art. 15 may be much simplified if the axes of coordinates can be chosen so as to be principal axes at the origin. In this case the expression takes the simple form
$$I = A\alpha^2 + B\beta^2 + C\gamma^2.$$

A method will presently be given by which we can always find these axes, but in some simpler cases we may determine their position by inspection. Let the body be symmetrical about the plane of xy. Then for every element m on one side of the plane whose coordinates are (x, y, z) there is another element of equal mass on the other side whose coordinates are $(x, y, -z)$. Hence for such a body $\Sigma mxz = 0$ and $\Sigma myz = 0$. If the body be a lamina in the plane of xy, then the z of every element is zero, and we have again $\Sigma mxz = 0$, $\Sigma myz = 0$.

Recurring to the table in Art. 8, we see that in every case the axes, about which the moments of inertia are given, are principal axes. Thus in the case of the ellipsoid, the three principal sections are all planes of symmetry, and therefore, by what has just been said, the principal diameters are principal axes of inertia. In applying the fundamental formula of Art. 15 to any body mentioned in the table, we may therefore always use the modified form given in this article.

17. **Examples.** *Let us now consider how the two important propositions of Arts.* 13 *and* 15 *are to be applied in practice.*

Ex. 1. Suppose we want the moment of inertia of an elliptic area of mass M and semi-axes a and b about a diameter making an angle θ with the major axis. The moments of inertia about the axes of a and b respectively are $\frac{1}{4}Mb^2$ and $\frac{1}{4}Ma^2$. By Art. 16 the moment of inertia about the diameter is $\frac{1}{4}Mb^2\cos^2\theta + \frac{1}{4}Ma^2\sin^2\theta$. If r be the length of the diameter this is known from the equation of the ellipse to be the same as $\dfrac{M}{4}\dfrac{a^2b^2}{r^2}$, which is a very convenient form in practice.

Ex. 2. Suppose we want the moment of inertia of the same ellipse about a tangent. Let p be the perpendicular from the centre on the tangent, then by Art. 13, the required moment is equal to the moment of inertia about a parallel axis through the centre together with $Mp^2 = \dfrac{M}{4}\dfrac{a^2b^2}{r^2} + Mp^2 = \dfrac{5M}{4}p^2$, since $pr = ab$.

Ex. 3. As an example of a different kind, let us find the moment of inertia of an ellipsoid of mass M and semi-axes (a, b, c) with regard to a diametral *plane* whose direction-cosines referred to the principal planes are (α, β, γ). By Art. 8, the moments of inertia with regard to the principal axes are $\frac{1}{5}M(b^2+c^2)$, $\frac{1}{5}M(c^2+a^2)$, $\frac{1}{5}M(a^2+b^2)$. Hence by Art. 5, the moments of inertia with regard to the principal *planes* are $\frac{1}{5}Ma^2$, $\frac{1}{5}Mb^2$, $\frac{1}{5}Mc^2$. Hence the required moment of inertia is $\frac{1}{5}M(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)$. If p be the perpendicular on the parallel tangent plane, we know by solid geometry that this is the same as $\frac{1}{5}Mp^2$.

Ex. 4. The moment of inertia of a rectangle whose sides are $2a$, $2b$ about a diagonal is $\dfrac{2M}{3}\dfrac{a^2b^2}{a^2+b^2}$.

Ex. 5. If k_1, k_2 be the radii of gyration of an elliptic lamina about two conjugate diameters, then $\dfrac{1}{k_1^2}+\dfrac{1}{k_2^2}=4\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

Ex. 6. The sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same.

Ex. 7. If M be the mass of a right cone, a its altitude and b the radius of the base, then the moment of inertia about the axis is $M\frac{3}{10}b^2$; that about a straight line through the vertex perpendicular to the axis is $M\frac{3}{5}(a^2+\frac{1}{4}b^2)$, that about a slant side $M\dfrac{3b^2}{20}\dfrac{6a^2+b^2}{a^2+b^2}$; that about a perpendicular to the axis through the centre of gravity is $M\frac{3}{80}(a^2+4b^2)$.

Ex. 8. If a be the altitude of a right cylinder, b the radius of the base, then the moment of inertia about the axis is $\frac{1}{2}Mb^2$ and that about a straight line through the centre of gravity perpendicular to the axis is $\frac{1}{4}M(\frac{1}{3}a^2+b^2)$.

Ex. 9. The moment of inertia of a body of mass M about a straight line whose equation is $\dfrac{x-f}{l}=\dfrac{y-g}{m}=\dfrac{z-h}{n}$ referred to any rectangular axes meeting at the centre of gravity is

$$Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm+M\{f^2+g^2+h^2-(fl+gm+hn)^2\},$$

where (l, m, n) are the direction-cosines of the straight line.

Ex. 10. The moment of inertia of an elliptic disc whose equation is

$$ax^2+2bxy+cy^2+2dx+2ey+1=0,$$

about a diameter parallel to the axis of x, is $\dfrac{M}{4}\cdot\dfrac{-Ha}{(ac-b^2)^2}$, where M is the mass and H is the determinant $ac-b^2+2bed-ae^2-cd^2$, usually called the discriminant.

Ex. 11. The moment of inertia of the elliptic disc whose equation in areal coordinates is $\phi(x, y, z)=0$ about a diameter parallel to the side a is

$$-M\left(\frac{\Delta}{a}\right)^2\frac{H}{2K^2}\left(\frac{d}{dy}-\frac{d}{dz}\right)^2\phi,$$

where Δ is the area, H the discriminant and K the bordered discriminant.

18. **Method of transformation of axes.** The method used in Art. 15 to find the moment of inertia about the straight line ON is really equivalent to a change of coordinate axes in which this straight line is taken as a new axis, say, of ξ, those of η and ζ not being required. We may now generalize this into a method which is often of great practical use.

Let us suppose that $\phi(\xi, \eta, \zeta)$ is any quadratic function, say

$$\phi=L_1\xi^2+L_2\eta^2+L_3\zeta^2+2K_1\eta\zeta+2K_2\zeta\xi+2K_3\xi\eta,$$

and that it is required to find $\Sigma m\phi(\xi, \eta, \zeta)$ the summation extending throughout any body.

	ξ	η	ζ
x	α	β	γ
y	α'	β'	γ'
z	α''	β''	γ''

Select some convenient set of axes which we may call x, y, z having the same origin such that the *six constants of the body*, viz. Σmx^2, Σmy^2, Σmz^2, Σmxy, Σmyz, Σmzx, are all known or can be easily found. Let the direction-cosines of these axes be given by the diagram in the margin.

We then have $\xi=\alpha x+\alpha' y+\alpha'' z$, $\eta=\beta x+\beta' y+\beta'' z$, $\zeta=\gamma x+\gamma' y+\gamma'' z$. Substituting these values and expanding we obtain an expression for $\Sigma m\phi(\xi, \eta, \zeta)$ in terms of the six known constants of the body.

The result may appear at first sight to be rather complicated, but if the new axes be properly chosen it reduces in most cases to a few terms. Thus if the *axes of* (x, y, z) *are principal axes all the terms* Σmxy, Σmyz, Σmzx *are zero.* Supposing this choice to be made, the formula reduces to the convenient form

$$\Sigma m\phi(\xi, \eta, \zeta) = \phi(\alpha, \beta, \gamma)\,\Sigma mx^2 + \phi(\alpha', \beta', \gamma')\,\Sigma my^2 + \phi(\alpha'', \beta'', \gamma'')\,\Sigma mz^2 \ldots (1).$$

In using this formula, the coefficient of Σmx^2 is obtained by substituting for (ξ, η, ζ) in $\phi(\xi, \eta, \zeta)$ the direction-cosines of the *new axis of* x, i.e. the cosines in the row of the diagram marked x. The coefficient of Σmy^2 may be obtained by substituting the direction-cosines of the new axis of y, i.e. the cosines in the row marked y, and so on.

If it be required to change the origin of coordinates also, this may be done by an application of the theorem in Art. 14.

If the body is a triangular area or a tetrahedral volume, the value of the integral $\Sigma m\phi$ may be written down at sight when the coordinates of the corners of the body are given. *We have merely to replace the body by any convenient system of equimomental points*, see Art. 36.

Ex. 1. The coordinates of the centre of an elliptic area are (f, g, h) and the direction-cosines of its axes are (α, β, γ) $(\alpha', \beta', \gamma')$, prove that

$$\Sigma m\zeta^2 = M(h^2 + \tfrac{1}{4}a^2\gamma^2 + \tfrac{1}{4}b^2\gamma'^2).$$

Ex. 2. Let Ox, Oy, Oz be the principal axes at the origin, prove that the product of inertia $F' = \Sigma m\xi\eta$ about two rectangular axes $O\xi$, $O\eta$ whose directions are $(\alpha, \alpha', \alpha'')$ (β, β', β'') is given by either of the formulae

$$\begin{aligned}\Sigma m\xi\eta &= \alpha\beta\Sigma mx^2 + \alpha'\beta'\Sigma my^2 + \alpha''\beta''\Sigma mz^2\\ &= -\alpha\beta A - \alpha'\beta' B - \alpha''\beta'' C.\end{aligned}$$

The first result is seen *at once* to be true by substituting the values of ξ, η given above; and the second result follows immediately from the first since $\alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0$. *These are very simple formulae to find products of inertia.*

Ex. 3. Let $(\gamma, \gamma', \gamma'')$ be the direction-cosines of a fixed axis $O\zeta$. Then as $O\xi$, $O\eta$ turn round $O\zeta$, prove that both $D'^2 + E'^2$ and $A'B' - F'^2$ are constant where A', B', C', D', E', F' are the moments and products of inertia of the body referred to these moving axes.

For by Ex. 2, $-D' = A\beta\gamma + B\beta'\gamma' + C\beta''\gamma''$, $\quad -E' = A\alpha\gamma + B\alpha'\gamma' + C\alpha''\gamma''$;

$$\therefore\ D'^2 + E'^2 = A^2\gamma^2(\alpha^2 + \beta^2) + 2AB\gamma\gamma'(\alpha\alpha' + \beta\beta') + \&c.;$$

since $\alpha^2 + \beta^2 = 1 - \gamma^2 = \gamma'^2 + \gamma''^2$ and $\alpha\alpha' + \beta\beta' = -\gamma\gamma'$ we have

$$D'^2 + E'^2 = (A - B)^2(\gamma\gamma')^2 + (B - C)^2(\gamma'\gamma'')^2 + (C - A)^2(\gamma''\gamma)^2.$$

Similarly $\qquad A'B' - F'^2 = BC\gamma^2 + CA\gamma'^2 + AB\gamma''^2.$

19. **The Ellipsoids of Inertia.** The expression which has been found in Art. 15 for the moment of inertia I about a straight line whose direction-cosines are (α, β, γ),

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

admits of a very useful geometrical interpretation.

Let a radius vector OQ move in any manner about the given point O, and be of such length that the moment of inertia about OQ may be *proportional* to the inverse square of the length. Then if R represent the length of the radius vector whose

direction-cosines are (α, β, γ), we have $I = M\epsilon^4/R^2$, where ϵ is some constant introduced to keep the dimensions correct, and M is the mass. We shall sometimes abbreviate $M\epsilon^4$ into the single symbol K. Hence the polar equation of the locus of Q is

$$\frac{K}{R^2} = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

Transforming to Cartesian coordinates, we have

$$K = AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY,$$

which is the equation of a quadric. Thus to every point O of a material body there is a corresponding quadric which possesses the property that the moment of inertia about any radius vector is represented by the inverse square of that radius vector. The convenience of this construction is, that the relations which exist between the moments of inertia about straight lines meeting at any given point may be discovered by help of the known properties of a quadric.

Since a moment of inertia is essentially positive, being by definition the sum of a number of squares, it is clear that every radius vector R must be real. Hence the quadric is always an ellipsoid. It is called the *momental ellipsoid,* and was first used by Cauchy, *Exercices de Math.* Vol. II.

So much has been written on the ellipsoids of inertia that it is difficult to determine what is really due to each of the various authors. The reader will find much information on these points in Prof. Cayley's report to the British Association on the *Special Problems of Dynamics*, 1862.

20. **The Invariants.** The momental ellipsoid is defined by a *geometrical* property, viz. that any radius vector is equal to some constant divided by the square root of the moment of inertia about that radius vector. Hence whatever coordinate axes are taken, we must always arrive at the same ellipsoid. If therefore the momental ellipsoid be referred to any set of rectangular axes, the coefficients of X^2, Y^2, Z^2, $-2YZ$, $-2ZX$, $-2XY$ in its equation will still represent the moments and products of inertia about these axes.

Since the discriminating cubic determines the lengths of the axes of the ellipsoid, it follows that its coefficients are unaltered by a transformation of axes. But these coefficients are

$$A + B + C,$$
$$AB + BC + CA - D^2 - E^2 - F^2,$$
$$ABC - 2DEF - AD^2 - BE^2 - CF^2.$$

Hence *for all rectangular axes having the same origin, these are invariable and all are greater than zero.*

21. It should be noticed that the constant ϵ is arbitrary, though when once chosen it cannot be altered. Thus we have a series of similar and similarly situated ellipsoids, any one of which may be used as a momental ellipsoid.

When the body is a plane lamina, a section of the ellipsoid corresponding to any point in the lamina by the plane of the lamina, is called a *momental ellipse* at that point.

If principal axes at any point O of a body be taken as axes of coordinates, the equation of the momental ellipsoid takes the simple form $AX^2 + BY^2 + CZ^2 = M\epsilon^4$, where M is the mass and ϵ^4 any constant. Let us now apply this to some simple cases.

Ex. 1. *To find the momental ellipsoid at the centre of a material elliptic disc.* Taking the same notation as before, we have $A = \frac{1}{4}Mb^2$, $B = \frac{1}{4}Ma^2$, $C = \frac{1}{4}M(a^2 + b^2)$. Hence the ellipsoid is $\frac{1}{4}Mb^2X^2 + \frac{1}{4}Ma^2Y^2 + \frac{1}{4}M(a^2 + b^2)Z^2 = M\epsilon^4$.

Since ϵ is any constant, this may be written $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} + \left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)Z^2 = \epsilon'$.

When $Z = 0$, this becomes an ellipse similar to the boundary of given disc. Hence we infer that the momental ellipse at the centre of an elliptic area is any similar and similarly situated ellipse. This also follows from Art. 17, Ex. 1.

Ex. 2. *To find the momental ellipsoid at any point O of a material straight rod AB of mass M and length $2a$.* Let the straight line OAB be the axis of x, O the origin, G the middle point of AB, $OG = c$. If the material line can be regarded as indefinitely thin, $A = 0$, $B = M(\frac{1}{3}a^2 + c^2) = C$, hence the momental ellipsoid is $Y^2 + Z^2 = \epsilon'^2$, where ϵ' is any constant. The momental ellipsoid is therefore an elongated spheroid, which becomes a right cylinder having the straight line for axis, when the rod becomes indefinitely thin.

Ex. 3. The momental ellipsoid at the centre of a material ellipsoid is

$$(b^2 + c^2)X^2 + (c^2 + a^2)Y^2 + (a^2 + b^2)Z^2 = \epsilon^4,$$

where ϵ is any constant. It should be noticed that *the longest and shortest axes of the momental ellipsoid coincide in direction with the longest and shortest axes respectively of the material ellipsoid.*

22. Conversely, we may show that *any ellipsoid being given, a real material body can be found of which it is the momental ellipsoid provided the sum of the squares of the reciprocals of any two of its axes is greater than the square of the reciprocal of the third.*

For let the moments of inertia about the principal diameters be $A = K/a^2$, $B = K/b^2$, $C = K/c^2$, then by Art. 5 it is necessary that the sum of any two of the three A, B, C should be greater than the third. Again, this condition is sufficient, for if we place two particles on each principal diameter, at such distances from the origin, $\pm p$, $\pm q$, $\pm r$, and of such masses, m, m', m'', that

$$4mp^2 = B + C - A, \quad 4mq^2 = C + A - B, \quad 4mr^2 = A + B - C,$$

these six particles will have the principal diameters for principal axes, and the given quantities, A, B, C, for their principal moments of inertia.

23. **Elementary Properties of Principal Axes.** By a consideration of some simple properties of ellipsoids, the following propositions are evident:

I. *Of the moments of inertia of a body about axes meeting at a given point, the moment of inertia about one of the principal axes is greatest and about another least.*

For, in the momental ellipsoid, the moment of inertia about a radius vector from the centre is least when that radius vector is greatest and *vice versâ*. And it is evident that the greatest and least radii vectores are two of the principal diameters.

It follows by Art. 5 that of the moments of inertia with regard to all *planes* passing through a given point, that with regard to one principal plane is greatest and with regard to another is least.

II. *If the three principal moments at any point O are equal to each other, the ellipsoid becomes a sphere.* Every diameter is then a principal diameter, and the radii vectores are all equal. Hence *every straight line through O is a principal axis at O, and the moments of inertia about them are all equal.*

For example, the perpendiculars from the centre of gravity of a cube on the three faces are principal axes; for, the body being referred to them as axes, we clearly have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. Also the three moments of inertia about them are by symmetry equal. Hence every axis through the centre of gravity of a cube is a principal axis, and the moments of inertia about them are all equal.

Next suppose the body to be a regular solid. Consider two planes drawn through the centre of gravity each parallel to a face of the solid. The relations of these two planes to the solid are in all respects the same. Hence also the momental ellipsoid at the centre of gravity must be similarly situated with regard to each of these planes, and the same is true for planes parallel to all the faces. Hence the ellipsoid must be a sphere and the moment of inertia will be the same about every axis.

Ex. 1. Three equal particles A, B, C are placed at the corners of an equilateral triangle; prove that the momental ellipse at their centre of gravity G is a circle.

By symmetry the diameters GA, GB, GC of the momental ellipse at G must be equal. The ellipse is therefore a circle.

Ex. 2. Four equal particles are placed at the corners of a tetrahedron. If the momental ellipsoid at their centre of gravity is a sphere, prove that the tetrahedron is regular.

Ex. 3. Any point O in a body being given and any plane drawn through it, prove that two straight lines at right angles can be drawn in this plane through O such that the product of inertia about them is zero.

These are the axes of the section of the momental ellipsoid at the point O formed by the given plane.

24. *At every point of a material system there are always three principal axes at right angles to each other.*

Construct the momental ellipsoid at the given point. Then it has been shown that the products of inertia about the axes are

half the coefficients of $-XY$, $-YZ$, $-ZX$ in the equation of the momental ellipsoid referred to these straight lines as axes of coordinates. Now if an ellipsoid be referred to its principal diameters as axes, these coefficients vanish. Hence the principal diameters of the ellipsoid are the principal axes of the system. But every ellipsoid has at least three principal diameters, hence every material system has at least three principal axes.

25. Ex. 1. The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point (p, q, r) is

$$\left(\frac{A}{M}+q^2+r^2\right)X^2+\left(\frac{B}{M}+r^2+p^2\right)Y^2+\left(\frac{C}{M}+p^2+q^2\right)Z^2-2qrYZ-2rpZX-2pqXY=\epsilon^4,$$

when referred to its centre as origin.

Ex. 2. Show that the cubic equation to find the three principal moments of inertia at any point (p, q, r) may be written in the form of a determinant

$$\begin{vmatrix} \frac{I-A}{M}-q^2-r^2 & pq & rp \\ pq & \frac{I-B}{M}-r^2-p^2 & qr \\ rp & qr & \frac{I-C}{M}-p^2-q^2 \end{vmatrix}=0.$$

If (l, m, n) be proportional to the direction-cosines of the axes corresponding to any one of the values of I, their values may be found from the equations

$$\left.\begin{aligned} &\{I-(A+Mq^2+Mr^2)\}\,l+Mpqm+Mrpn=0,\\ &Mpql+\{I-(B+Mr^2+Mp^2)\}\,m+Mqrn=0,\\ &Mrpl+Mqrm+\{I-(C+Mp^2+Mq^2)\}\,n=0. \end{aligned}\right\}$$

Thus (l, m, n) are proportional to the minors of the constituents of any row of the determinant.

Ex. 3. If $S=0$ be the equation to the momental ellipsoid at the centre of gravity O referred to any rectangular axes written in the form given in Art. 19, then the momental ellipsoid at the point P whose coordinates are (p, q, r) is

$$S+M(p^2+q^2+r^2)(X^2+Y^2+Z^2)-M(pX+qY+rZ)^2=0.$$

Hence show (1) that the conjugate planes of the straight line OP in the momental ellipsoids at O and P are parallel and (2) that the sections perpendicular to OP have their axes parallel.

26. **Ellipsoid of Gyration.** The reciprocal surface of the momental ellipsoid is another ellipsoid, which has also been employed to represent, geometrically, the positions of the principal axes and the moment of inertia about any line.

We shall require the following elementary proposition. The reciprocal surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the ellipsoid $a^2x^2+b^2y^2+c^2z^2=\epsilon^4$.

Let ON be the perpendicular from the origin O on the tangent plane at any point P of the first ellipsoid, and let l, m, n be the direction-cosines of ON, then $ON^2=a^2l^2+b^2m^2+c^2n^2$. Produce ON to Q so that $OQ=\epsilon^2/ON$, then Q is a point on the reciprocal surface. Let $OQ=R$; $\therefore \epsilon^4=(a^2l^2+b^2m^2+c^2n^2)R^2$. Changing this to rectangular coordinates, we get $\epsilon^4=a^2x^2+b^2y^2+c^2z^2$.

To each point of a material body there corresponds a series of similar momental ellipsoids. If we reciprocate these we get another series of similar ellipsoids coaxial with the first, and such that the moments of inertia of the body about the perpendiculars on the tangent planes to any one ellipsoid are proportional to the squares of those perpendiculars. It is, however, convenient to call *that particular ellipsoid the ellipsoid of gyration which makes the moment of inertia about a perpendicular on a tangent plane equal to the product of the mass into the square of that perpendicular*. If M be the mass of the body and A, B, C the principal moments, the equation of the ellipsoid of gyration is

$$\frac{X^2}{A}+\frac{Y^2}{B}+\frac{Z^2}{C}=\frac{1}{M}.$$

It is clear that the constant on the right-hand side must be $1/M$, for when Y and Z are put equal to zero, MX^2 must by definition be A.

27. Conversely, the series of momental ellipsoids at any point of a body may be regarded as the reciprocals, with different constants, of the ellipsoid of gyration at that point. They are all of an opposite shape to the ellipsoid of gyration, having their longest axes in the direction of the shortest axis and their shortest axes in the direction of the longest axis of the ellipsoid of gyration. The momental ellipsoids however resemble the general shape of the body more nearly than the ellipsoid of gyration. They are protuberant where the body is protuberant and compressed where the body is compressed. The exact reverse of this is the case in the ellipsoid of gyration. See Art. 22, Ex. 3.

28. Ex. 1. To find the ellipsoid of gyration at the centre of a material elliptic disc. Taking the values of A, B, C given in Art. 22, Ex. 1, we see that the ellipsoid of gyration is $\dfrac{X^2}{b^2}+\dfrac{Y^2}{a^2}+\dfrac{Z^2}{a^2+b^2}=\dfrac{1}{4}$.

Ex. 2. The ellipsoid of gyration at any point O of a material rod AB is $\dfrac{X^2}{0}+\dfrac{Y^2}{\frac{1}{3}a^2+c^2}+\dfrac{Z^2}{\frac{1}{3}a^2+c^2}=1$, taking the notation of Art. 21, Ex. 2. It is thus a very flat spheroid which, when the rod is indefinitely thin, becomes a circular area, whose centre is at O, whose radius is $\sqrt{\frac{1}{3}a^2+c^2}$ and whose plane is perpendicular to the rod.

Ex. 3. It may be shown that the general equation of the ellipsoid of gyration referred to any set of rectangular axes meeting at the given point of the body is

$$\begin{vmatrix} A & -F & -E & MX \\ -F & B & -D & MY \\ -E & -D & C & MZ \\ MX & MY & MZ & M \end{vmatrix}=0,$$

or, when expanded,

$$(BC-D^2)X^2+(CA-E^2)Y^2+(AB-F^2)Z^2+2(AD+EF)YZ+2(BE+FD)ZX$$
$$+2(CF+DE)XY=\frac{1}{M}(ABC-AD^2-BE^2-CF^2-2DEF).$$

The right-hand side, when multiplied by M, is the discriminant obtained by leaving out the last row and the last column, and the coefficients of X^2, Y^2, Z^2, $2ZX$, $2XY$, $2YZ$ are the minors of this discriminant.

29. The use of the ellipsoid whose equation referred to the principal axes at the centre of gravity is

$$\frac{X^2}{\Sigma mx^2} + \frac{Y^2}{\Sigma my^2} + \frac{Z^2}{\Sigma mz^2} = \frac{5}{M},$$

has been suggested by Legendre in his *Fonctions Elliptiques.* This ellipsoid is to be regarded as a homogeneous solid of such density that its mass is equal to that of the body. By Art. 8, Ex. 3, it possesses the property that its moments of inertia with regard to its principal axes, and therefore by Art. 15 its moments of inertia with regard to all planes and axes, are the same as those of the body. We may call this ellipsoid the *equimomental* ellipsoid or *Legendre's* ellipsoid.

Ex. If a plane move so that the moment of inertia with regard to it is always proportional to the square of the perpendicular from the centre of gravity on the plane, then this plane envelopes an ellipsoid similar to Legendre's ellipsoid.

30. There is another ellipsoid which is sometimes used. By Art. 15 the moment of inertia with reference to a plane whose direction-cosines are (α, β, γ) is

$$I' = \Sigma mx^2 . \alpha^2 + \Sigma my^2 . \beta^2 + \Sigma mz^2 . \gamma^2 + 2\Sigma myz . \beta\gamma + 2\Sigma mzx . \gamma\alpha + 2\Sigma mxy . \alpha\beta.$$

Hence, as in Art. 19, we may construct the ellipsoid

$$\Sigma mx^2 . X^2 + \Sigma my^2 . Y^2 + \Sigma mz^2 . Z^2 + 2\Sigma myz . YZ + 2\Sigma mzx . ZX + 2\Sigma mxy . XY = K.$$

Then the moment of inertia with regard to any plane through the centre is represented by the inverse square of the radius vector perpendicular to that plane.

If we compare the equation of the momental ellipsoid with that of this ellipsoid, we see that one may be obtained from the other by subtracting the same quantity from each of the coefficients of X^2, Y^2, Z^2. Hence the two ellipsoids have their circular sections coincident in direction.

This ellipsoid may also be used to find the moments of inertia about any straight line through the origin. For we may deduce from Art. 15 that the moment of inertia about any radius vector is represented by the difference between the inverse square of that radius vector and the sum of the inverse squares of the semi-axes. This ellipsoid is a reciprocal of Legendre's ellipsoid. All these ellipsoids have their principal diameters coincident in direction, and any one of them may be used to determine the directions of the principal axes at any point.

31. When the body considered is a lamina, the section of the ellipsoid of gyration at any point of the lamina by the plane of the lamina is called the *ellipse of gyration.* If the plane of the lamina be the plane of xy, we have $\Sigma mz^2 = 0$. The section of the fourth ellipsoid is then clearly the same as an ellipse of gyration at the point. If any momental ellipse be turned round its centre through a right angle it evidently becomes similar and similarly situated to the ellipse of gyration. Thus, in the case of a lamina, any one of these ellipses may be easily changed into the others.

32. **Equimomental Cone.** *A straight line passes through a fixed point O and moves about it in such a manner that the moment of inertia about the line is always the same and equal to a given quantity I. To find the equation of the cone generated by the straight line.*

Let the principal axes at O be taken as the axes of coordinates, and let (α, β, γ) be the direction-cosines of the straight line in any position. Then by Art. 16 we have $A\alpha^2 + B\beta^2 + C\gamma^2 = I$. Hence the equation of the locus is

$$(A - I)\,\alpha^2 + (B - I)\,\beta^2 + (C - I)\,\gamma^2 = 0,$$

or, transforming to Cartesian coordinates,

$$(A-I)x^2+(B-I)y^2+(C-I)z^2=0.$$

It appears from this equation that the principal diameters of the cone are the principal axes of the body at the given point.

The given quantity I must be less than the greatest and greater than the least of the moments A, B, C. Let A, B, C be arranged in descending order of magnitude; then if I be less than B, the cone has its concavity turned towards the axis C, if I be greater than B the concavity is turned towards the axis A, if $I=B$ the cone becomes two planes which are coincident with the central circular sections of the momental ellipsoid at the point O.

The geometrical peculiarity of this cone is that its circular sections in all cases are coincident in direction with the circular sections of the momental ellipsoid at the vertex.

This cone is called an *equimomental cone* at the point at which its vertex is situated.

33. **On Equimomental Bodies.** Two bodies or systems of bodies are said to be equimomental when their moments of inertia about all straight lines are equal each to each.

34. If two systems have the same centre of gravity, the same mass, the same principal axes and principal moments at the centre of gravity, it follows from the two fundamental propositions of Arts. 13 and 15 that their moments of inertia about all straight lines are equal, each to each.

The converse theorem is also true. If the two bodies have equal moments of inertia about every straight line, it is evident that the axes of maxima and minima moments are the same in the two bodies. Of all straight lines having a given direction that one has the least moment of inertia for either body which passes through the centre of gravity of that body (Art. 13). Consider any direction perpendicular to the straight line joining the two centres of gravity G, G'. The minimum for one body passes through G and for the other through G'. They cannot be the same unless G, G' coincide.

Next consider all the directions which pass through the common centre of gravity. The axes of greatest and least moments of inertia for each body are two of the principal axes of that body (Art. 23). These must therefore coincide in the two bodies. The third axis in each body is perpendicular to these two, and they also must coincide.

Lastly, consider two parallel axes at a distance p apart, one passing through the common centre of gravity. By the theorem of parallel axes, the difference of the moments of inertia about these for either body is Mp^2, where M is the mass of that body. But both the moments of inertia and the distance p are the same for each body. Hence the masses are also equal.

It is easy to see that two equimomental systems must have

the same momental ellipsoid, and therefore the same principal axes at every point.

35. **Case of a Triangle.** *To find the moments and products of inertia of a triangle about any axes whatever.*

If β and γ be the distances of the angular points B, C of a triangle ABC from any straight line AX drawn through the angle A, in the plane of the triangle, it is known that the moment of inertia of the triangle about AX is $\frac{1}{6}M(\beta^2+\beta\gamma+\gamma^2)$, where M is the mass of the triangle.

Let three equal particles, the mass of each being $\frac{1}{3}M$, be placed at the middle points of the three sides. Then it is easily seen, that the moment of inertia of the three particles about AX is

$$\frac{M}{3}\left\{\left(\frac{\beta+\gamma}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2+\left(\frac{\beta}{2}\right)^2\right\},$$

which is the same as that of the triangle. The three particles, treated as one system, and the triangle have the same centre of gravity. Let this point be called O. Draw any straight line OX' through the common centre of gravity O parallel to AX, then it is evident that the moments of inertia of the two systems about OX' are also equal.

Since this equality exists for all straight lines through O in the plane of the triangle, it will be true for two straight lines OX', OY' at right angles, and therefore also for a straight line OZ' perpendicular to the plane of the triangle.

One of the principal axes at O of the triangle, and of the systems of three particles, is normal to the plane, and therefore the same for the two systems. The principal axes at O in the plane, are those two straight lines about which the moments of inertia are greatest and least, and therefore by what precedes these axes are the same for the two systems. If at any point two systems have the same principal axes and principal moments, they have also the same moments of inertia about all axes through that point, and the same products of inertia about any two straight lines meeting in that point. And if this point be the centre of gravity of both systems, the same thing will also be true for any other point.

If then a particle whose mass is one-third that of the triangle be placed at the middle point of each side, the moment of inertia of the triangle about any straight line, is the same as that of the system of particles, and the product of inertia about any two straight lines meeting one another, is the same as that of the system of particles.

36. The existence of equimomental points is of the greatest utility in finding the moments and products of inertia of a body about any axes. *They may also be used for more general integrations.*

Thus suppose any given body to be equimomental to three particles whose coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). Since the masses placed at these points may not in all cases be equal, let these masses be respectively M_1, M_2, M_3, where of course the sum is equal to the mass of the body. Let $\phi(x, y, z)$ be any function of x, y, z which does not contain any power higher than the second. Let it be required to find the value of the integral or sum $\Sigma m\phi(x, y, z)$ taken throughout the body, where m is an element of the mass. The required integral is evidently equal to $M_1\phi(x_1, y_1, z_1) + M_2\phi(x_2, y_2, z_2) + M_3\phi(x_3, y_3, z_3)$.

By properly choosing the equivalent points we may use a similar rule in which ϕ is any *cubic* or *quartic* function of x, y, z, but as these cases are not wanted in rigid dynamics we shall merely state a few results a little farther on.

The same body may be equimomental to several systems of points, and some of these sets may be more convenient than the others. In order that a set of equimomental points may be useful it is necessary (1) that the points should be so conveniently placed in the body that their coordinates can be easily found with regard to any given axes, (2) that the number of points employed in the set should be as small as possible. Of these two requisites the first is by far the more important.

Equimomental points have another use besides that of shortening integrations which may otherwise be troublesome. It will be presently seen that they have a dynamical importance.

37. *A momental ellipsoid at the centre of gravity of any triangle may be found as follows.*

Let an ellipse be inscribed in the triangle touching two of the sides AB, BC in their middle points F, D. Then, by Carnot's theorem, it touches the third side CA in its middle point E. Since DF is parallel to CA the tangent at E, the straight line joining E to the middle point N of DF passes through the centre, and therefore the centre of the conic is at O the centre of gravity of the triangle.

This conic may be shown to be a momental ellipse of the triangle at O. To prove this, let us find the moment of inertia of the triangle about OE. Let $OE = r$, and let r' be the semi-conjugate diameter, and ω the angle between r and r'. Now $ON = \frac{1}{2}r$, and hence from the equation of the ellipse $FN^2 = \frac{3}{4}r'^2$,

$$\left.\begin{matrix}\text{therefore moment of}\\ \text{inertia about } OE\end{matrix}\right\} = \tfrac{2}{3}M \cdot \tfrac{3}{4}r'^2 \sin^2\omega, \quad = \frac{M}{2} \cdot \frac{\Delta'^2}{\pi^2 r^2};$$

where Δ' is the area of the ellipse, so that the moments of inertia of the system about OE, OF, OD are proportional inversely to OE^2, OF^2, OD^2. If we take a momental ellipse of the right dimensions, it will cut the inscribed conic in E, F, and D, and therefore also at the opposite ends of the diameters through these points. But two conics cannot cut each other in six points unless they are identical. Hence this conic is a momental ellipse at O of the triangle.

A normal at O to the plane of the triangle is a principal axis of the triangle (Art. 16). Hence a momental ellipsoid of the triangle has the inscribed conic for one principal section. If $2a$ and $2b$ be the lengths of the axes of this conic, $2c$ that of the axis of the ellipsoid which is perpendicular to the plane of the lamina, we have, by Arts. 7 and 19, $1/c^2 = 1/a^2 + 1/b^2$.

If the triangle be an equilateral triangle, the momental ellipsoid becomes a spheroid, and every axis through the centre of gravity in the plane of the triangle is a principal axis.

Since any similar and similarly situated ellipse is also a momental ellipse, we may take the ellipse circumscribing the triangle, and having its centre at the centre of gravity, as the momental ellipse of the triangle.

38. Ex. 1. A momental ellipse at an angular point of a triangular area touches the opposite side at its middle point and bisects the adjacent sides.

Ex. 2. A momental ellipse at the middle point F of the side AB of a triangular lamina ABC circumscribes the triangle and has FC, FB for conjugate diameters. Prove also that another momental ellipse at the same point F touches the sides AC, BC at their middle points.

Ex. 3. The principal radii of gyration at the centre of gravity of a triangle are the roots of the equation
$$x^4 - \frac{a^2+b^2+c^2}{36}x^2 + \frac{\Delta^2}{108} = 0,$$
where Δ is the area of the triangle.

Ex. 4. The direction of the principal axes at the centre of gravity O of a triangle may be constructed thus. Draw at the middle point D of any side BC lengths $DH = \frac{6k^2}{p}$, $DH' = \frac{6k'^2}{p}$ along the perpendicular, where p is the perpendicular from A on BC and k, k' are the principal radii of gyration found by the last example. Then OH, OH' are the directions of the principal axes at O, whose moments of inertia are respectively Mk^2 and Mk'^2.

Ex. 5. The directions of the principal axes and the principal moments at the centre of gravity may also be determined thus. Draw at the middle point D of any side BC a perpendicular $DK = BC/2\sqrt{3}$. Describe a circle on OK as diameter and join D to the middle point of OK by a line cutting the circle in R and S, then OR, OS are the directions of the principal axes, and the moments of inertia about them are respectively $\frac{1}{2}M \,.\, DS^2$ and $\frac{1}{2}M \,.\, DR^2$.

Ex. 6. Let four particles each one-sixth of the mass of the area of a parallelogram be placed at the middle points of the sides and a fifth particle one-third of the same mass at the centre of gravity, *then these five particles and the area of the parallelogram are equimomental systems.*

Ex. 7. Let particles each equal to one-twelfth of the mass of a quadrilateral area be placed at each corner and let a fifth particle of negative mass but also one-twelfth be placed at the intersection of the diagonals. Then the centre of gravity of the quadrilateral area is the centre of gravity of these five particles. Let a sixth particle equal to three-quarters of the mass of the quadrilateral be placed at the centre of gravity thus found. Prove that *these six particles are equimomental to the quadrilateral area.*

Ex. 8. Let particles each equal to one-quarter of the mass of an elliptic area be placed at the middle points of the chords joining the extremities of any pair of conjugate diameters. Prove that *these four particles are equimomental to the elliptic area.*

Ex. 9. Let a tenth of the mass of a solid homogeneous ellipsoid be placed at each of the six extremities of a set of conjugate diameters and two-fifths of the mass at the centre, prove that *this system of particles is equimomental to the ellipsoid.*

Ex. 10. Any sphere of radius a and mass M is equimomental to a system of four particles each of mass $\frac{3M}{20}\left(\frac{a}{r}\right)^2$ placed so that their distances from the centre make equal angles with each other and are each equal to r, and a fifth particle equal to the remainder of the mass of the sphere placed at the centre.

39. **Case of a Tetrahedron.** *To find the moments and products of inertia of a tetrahedron about any axes whatever, i.e. to find a system of equimomental particles.*

Let $ABCD$ be the tetrahedron. Through one angular point D draw any plane and let it be taken as the plane of xy. Let D be the area of the base ABC, α, β, γ the distances of its angular points from the plane of xy, and p the length of the perpendicular from D on the base ABC.

Let PQR be any section parallel to the base ABC and of thickness du, where u is the perpendicular from D on PQR. The moment of inertia of the triangle PQR with respect to the *plane* of xy is the same as that of three equal particles, each one-third its mass, placed at the middle points of its sides. The volume of the element $PQR = \frac{u^2}{p^2} Ddu$. The ordinates of the middle points of the sides AB, BC, CA are respectively $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$. Hence, by similar triangles, the ordinates of the middle points of PQ, QR, RP are $\frac{1}{2}(\alpha+\beta)\,u/p$, $\frac{1}{2}(\beta+\gamma)\,u/p$, $\frac{1}{2}(\gamma+\alpha)\,u/p$.

The moment of inertia of the triangle PQR with regard to the plane xy is therefore

$$\frac{1}{3}\frac{u^2}{p^2} Ddu \left\{\left(\frac{\beta+\gamma}{2}\frac{u}{p}\right)^2 + \left(\frac{\gamma+\alpha}{2}\frac{u}{p}\right)^2 + \left(\frac{\alpha+\beta}{2}\frac{u}{p}\right)^2\right\}.$$

Integrating from $u=0$ to $u=p$, we have the moment of inertia of the tetrahedron with regard to the plane xy

$$= \tfrac{1}{10} V \{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta\},$$

where V is the volume.

If particles each one-twentieth of the mass of the tetrahedron were placed at each of the angular points and the rest of the mass, viz. four-fifths, were collected at the centre of gravity, the moment of inertia of these five particles with regard to the plane of xy would be

$$= V\frac{4}{5}\left(\frac{\alpha+\beta+\gamma}{4}\right)^2 + \frac{V}{20}\alpha^2 + \frac{V}{20}\beta^2 + \frac{V}{20}\gamma^2,$$

which is the same as that of the tetrahedron.

The centre of gravity of these five particles is the centre of gravity of the tetrahedron, and together they make up the mass of the tetrahedron. Hence, by Art. 13, the moments of inertia of the two systems with regard to any plane through the centre of gravity are the same, and by the same article this equality will exist for all planes whatever. It follows, by Art. 5, that the moments of inertia about any straight line are also equal. *The two systems are therefore equimomental.*

40. **Theory of Projections.** If the distance of every point in a given figure in space from some fixed plane be increased in a

fixed ratio, the figure thus altered is called the *projection* of the given figure. By projecting a figure from three planes at right angles as base planes in succession, the figure may be often much simplified. Thus an ellipsoid can always be projected into a sphere, and any tetrahedron into a regular tetrahedron.

It is clear that if the base plane from which the figure is projected be moved parallel to itself into a position distant D from its former position, no change of form is produced in the projected figure. If n be the fixed ratio of projection the projected figure has merely been moved through a space nD perpendicular to the base plane. We may therefore suppose the base plane to pass through any given point which may be convenient.

41. *If two bodies are equimomental, their projections are also equimomental.*

Let the origin be the common centre of gravity, then the two bodies are such that $\Sigma m = \Sigma m'$; $\Sigma mx = 0$, $\Sigma m'x' = 0$, &c., $\Sigma mx^2 = \Sigma m'x'^2$, $\Sigma myz = \Sigma m'y'z'$, &c., unaccented letters referring to one body and accented letters to the other. Let both the bodies be projected from the plane of xy in the fixed ratio $1 : n$. Then any point whose coordinates are (x, y, z) is transferred to (x, y, nz) and (x', y', z') to (x', y', nz'). Also the elements of mass m, m' become nm and nm'. It is evident that the above equalities are not affected by these changes, and that therefore the projected bodies are equimomental.

The projection of a momental ellipse of a plane area is a momental ellipse of the projection.

Let the figure be projected from the axis of x as base line, so that any point (x, y) is transferred to (x, y') where $y' = ny$, and any element of area m becomes m' where $m' = nm$. Then

$$\Sigma mx^2 = \frac{1}{n}\Sigma m'x^2,\ \ \Sigma mxy = \frac{1}{n^2}\Sigma m'xy',\ \ \Sigma my^2 = \frac{1}{n^3}\Sigma m'y'^2.$$

The momental ellipses of the primitive and the projection are

$$\Sigma my^2X^2 - 2\Sigma mxyXY + \Sigma mx^2Y^2 = M\epsilon^4,$$

$$\Sigma m'y'^2X'^2 - 2\Sigma m'xy'X'Y' + \Sigma m'x^2Y'^2 = M'\epsilon'^4.$$

To project the former we put $X' = X$, $Y' = nY$. Its equation becomes identical with the latter by virtue of the above equalities when we put $\epsilon'^4 = \epsilon^4 n^2$.

42. Ex. 1. A momental ellipse of the area of a square at its centre of gravity is easily seen to be the inscribed circle. By projecting this figure first with one side as base line, and secondly with a diagonal as base, the square becomes successively a rectangle and a parallelogram. Hence one momental ellipse at the centre of gravity of a parallelogram is the inscribed conic touching the sides at their middle points.

Ex. 2. By projecting an equilateral triangle into any triangle, we may infer the results of some of the previous articles, but the method will be best explained by its application to a tetrahedron.

Ex. 3. Since any ellipsoid may be obtained by projecting a sphere, we infer by Art. 38, Ex. 10, that any solid ellipsoid of mass M is equimomental to a system of four particles each of mass $\frac{3M}{20}\frac{1}{n^2}$ placed on a similar ellipsoid whose linear dimensions are n times as great as those of the material ellipsoid, so that the eccentric lines of the particles make equal angles with each other, and a fifth particle equal to the remainder of the mass of the ellipsoid placed at the centre of gravity.

If this material ellipsoid be the Legendre's ellipsoid of any given body, we see that any body whatever is equimomental to a system of five particles placed as above described on an ellipsoid similar to the Legendre's ellipsoid of the body.

Ex. 4. Show that a solid oblique cone on an elliptic base is equimomental to a system of three particles each one-tenth of the mass of the cone placed on the circumference of the base so that the differences of their eccentric angles are equal, a fourth particle equal to three-tenths of the cone placed at the middle point of the straight line joining the vertex to the centre of gravity of the base, and a fifth particle to make up the mass of the cone placed at the centre of gravity of the volume.

43. *To find an ellipsoid equimomental to any tetrahedron.* The moments of inertia of a regular tetrahedron with regard to all *planes* through the centre of gravity O are equal by Art. 23. If r be the radius of the inscribed sphere, the moment with regard to a plane parallel to one face is easily seen by Art. 39 to be $M\frac{3r^2}{5}$. If then we describe a sphere of radius $\rho=\sqrt{3}r$, with its centre at the centre of gravity, and its mass equal to that of the tetrahedron, this sphere and the tetrahedron will be equimomental. Since the centre of gravity of any face projects into the centre of gravity of the projected face, we infer that the ellipsoid to which any tetrahedron is equimomental is similar and similarly situated to that inscribed in the tetrahedron and touching each face in its centre of gravity, but has its linear dimensions greater in the ratio $1:\sqrt{3}$. It may also be easily seen that the sphere whose radius is $\rho=\sqrt{3}r$, touches each edge of the regular tetrahedron at its middle point. Hence we infer that the ellipsoid equimomental to any tetrahedron touches each edge at its middle point and has its centre at the centre of gravity of the volume.

Ex. 1. If E^2 be the sum of the squares of the edges of a tetrahedron, F^2 the sum of the squares of the areas of the faces and V the volume, show that the semi-axes of the ellipsoid inscribed in the tetrahedron, touching each face in the centre of gravity and having its centre at the centre of gravity of the tetrahedron, are the roots of

$$\rho^6-\frac{E^2}{2^4.3}\rho^4+\frac{F^2}{2^4.3^2}\rho^2-\frac{V^2}{2^6.3}=0,$$

and that, if the roots be $\pm\rho_1$, $\pm\rho_2$, $\pm\rho_3$, the moments of inertia with regard to the principal *planes* of the tetrahedron are $M\frac{3\rho_1{}^2}{5}$, $M\frac{3\rho_2{}^2}{5}$, $M\frac{3\rho_3{}^2}{5}$.

Ex. 2. If a perpendicular EP be drawn at the centre of gravity E of any face $=4\rho^2/p$, where p is the perpendicular from the opposite corner of the tetrahedron on that face, then P is a point on the principal plane corresponding to the root ρ of the cubic.

44. *Four particles of equal mass can always be found which are equimomental to any given solid body.*

Let O be the centre of gravity of the body, Ox, Oy, Oz the principal axes at O. Let the moments of inertia with regard to the coordinate *planes* be Ma^2, $M\beta^2$, and $M\gamma^2$. By Art. 34, the mass of each particle must be $\frac{1}{4}M$. Let $(x_1y_1z_1)$ &c. $(x_4y_4z_4)$

be the required coordinates of these four points. Then these twelve coordinates must satisfy the nine equations

$$\Sigma x^2=4\alpha^2,\ \Sigma y^2=4\beta^2,\ \Sigma z^2=4\gamma^2,\ \Sigma xy=0,\ \Sigma yz=0,\ \Sigma zx=0,\ \Sigma x=0,\ \Sigma y=0,\ \Sigma z=0.$$

Now if we write $x_1=\alpha\xi_1$, $x_2=\alpha\xi_2$ &c. $y_1=\beta\eta_1$, $y_2=\beta\eta_2$ &c. $z_1=\gamma\zeta_1$ &c. we have nine equations to find the twelve coordinates $(\xi_1\eta_1\zeta_1)$ &c. $(\xi_4\eta_4\zeta_4)$ which differ from those just written down only in having α^2, β^2, γ^2 each replaced by unity. These modified equations express that the momental ellipsoid at O of the four particles must be a sphere. The equations are therefore satisfied if the four points, whose coordinates are represented by the *Greek letters*, are the corners of a regular tetrahedron. (See also Art. 23, Ex. 2.) This tetrahedron may be regarded as inscribed in a sphere whose radius is $\sqrt{3}$. If we project this sphere into an ellipsoid whose semi-axes are α, β, γ the regular tetrahedron will be deformed into an oblique tetrahedron. The corners of this oblique tetrahedron are the required equimomental points.

In the same way we may prove that three particles of equal mass can always be found which are equimomental to any plane area. If $M\alpha^2$, $M\beta^2$, and zero are the moments of inertia of the area about the principal *planes* at the centre of gravity, the result is that these particles must lie on the ellipse $\beta^2x^2+\alpha^2y^2=2\alpha^2\beta^2$. It also follows that, if one of these points, as D, be taken anywhere on this ellipse, the other two points, E and F, are at the opposite extremities of that chord which is bisected in some point N by the produced radius DO so that $ON=\frac{1}{2}OD$.

45. **Moments with Higher Powers.** These moments are not often wanted in dynamics though they are useful in other subjects. It will therefore be sufficient to state here some general results and to sketch the proofs in a note at the end of this volume. Some generalisations will also be added.

Let $d\sigma$ and dv be any elementary area and volume as the case may be. Let z be its ordinate referred to any plane of xy. Our object is to find the integral $\int z^n d\sigma$ or $\int z^n dv$ for a triangle, quadrilateral, tetrahedron, &c.

Let the coordinates of the corners of the body be $(x_1y_1z_1)$, $(x_2y_2z_2)$, &c. Let $S_n(z_1z_2$, &c.) represent the sum of the different homogeneous products of n dimensions of as many of the z's as are included in the bracket.

Then for a triangle of area Δ

$$\int z^n d\sigma=\frac{1\,.\,2\,.\,\Delta}{(n+1)(n+2)}\,.\,S_n(z_1z_2z_3).$$

For a quadrilateral of area Δ

$$\int z^n d\sigma=\frac{1\,.\,2\,.\,\Delta}{(n+1)(n+2)}\{S_n(z_1z_2z_3z_4)-z'S_{n-1}(z_1z_2z_3z_4)\},$$

where z' is the ordinate of the intersection of the diagonals.

For a tetrahedron of volume V

$$\int z^n dv=\frac{1\,.\,2\,.\,3\,.\,V}{(n+1)(n+2)(n+3)}S_n(z_1z_2z_3z_4).$$

For two tetrahedra joined together, whose united volume is V

$$\int z^n dv = \frac{1 \cdot 2 \cdot 3 \cdot V}{(n+1)(n+2)(n+3)} \{S_n(z_1 \ldots z_5) - z' S_{n-1}(z_1 \ldots z_5)\},$$

where z' is the ordinate of the point of intersection of the common base with the straight line joining the two vertices.

We notice that, except for the factor Δ or V representing the area or volume, these four expressions are functions of the ordinates only of the corners and are not functions of the differences of the abscissæ.

When the value of $\int z^n d\sigma$ is known that of $n \int x z^{n-1} d\sigma$ can be found by performing the operation $x_1 \dfrac{d}{dz_1} + x_2 \dfrac{d}{dz_2} + \ldots$ on the former result. The value of $n(n-1)\int x^2 z^{n-2} d\sigma$ can be found by repeating the operation and so on.

Lastly, it may be shown that when two bodies are such that the values of $\int z^n d\sigma$ are equal, each to each, for all planes of xy these bodies are equimomental.

Ex. 1. If $\phi(x, y, z)$ be a function not higher than the third degree the value of $\int \phi\, d\sigma$ for any triangle can be found by using seven equivalent or equimomental points. We collect *one-twentieth* of the mass of the area at each corner, *two-fifteenths* at the middle point of each side, and *the rest, viz. nine-twentieths*, at the centre of gravity.

Ex. 2. If $\phi(x, y, z)$ be not higher than the third degree the value of $\int \phi\, dv$ for a tetrahedron can be represented by eight equivalent points. We collect *nine-fortieths* of the volume at the centre of gravity of each face and *one-fortieth* at each corner. *Other examples may be found in No.* 83 *Quarterly Journal of Mathematics*, 1886.

46. **Theory of Inversion.** *To explain how the theory of inversion can be applied to find moments of inertia.*

Let a radius vector drawn from some fixed origin O to any point P of a figure be produced to P', where the rectangle $OP \cdot OP' = \kappa^2$, κ being some given quantity. Then as P travels all over the given figure, P' traces out another which is called the inverse of the given figure.

Let (x, y, z) be the coordinates of P, (x', y', z') those of P'; r, r' the radii vectores, dv, dv' corresponding polar elements of volume; ρ, ρ', dm, dm' their respective densities and masses. Let $d\omega$ be the solid angle subtended at O by either dv or dv'. Then

$$dv' = r'^2 d\omega\, dr' = \left(\frac{\kappa}{r}\right)^6 r^2 d\omega\, dr = \left(\frac{\kappa}{r}\right)^6 dv,$$

and since $\dfrac{x'}{r'} = \dfrac{x}{r}$ we have $x'^2 dv' = \left(\dfrac{\kappa}{r}\right)^{10} x^2 dv$. Now $dm = \rho\, dv$, $dm' = \rho' dv'$. If then we take $\dfrac{\rho'}{\rho} = \left(\dfrac{r}{\kappa}\right)^{10}$ we have $\Sigma x'^2 dm' = \Sigma x^2 dm$, with similar equalities in the case of all the other moments and products of inertia.

When the body is an area or an arc the ratio of dv' to dv is different. We have in these cases respectively $\dfrac{dv'}{dv} = \left(\dfrac{\kappa}{r}\right)^4$ or $\left(\dfrac{\kappa}{r}\right)^2$. Similar results however follow which may be all summed up in the following theorem.

THEOR. I. *Let any body be changed into another by inversion with regard to any point O. If the densities at corresponding points be denoted by* ρ, ρ' *and their distances from O by* r, r', *let* $\rho'=\rho\left(\frac{\kappa}{r'}\right)^n$. *Then these two bodies have the same moments of inertia with regard to all straight lines through O.* Here $n=10$, 8 or 6 according as the body is a *volume*, an *area* or an *arc*.

It also follows that the two bodies have the same principal axes at the point O, and the same ellipsoids of gyration.

We may also obtain the following theorem by the use of Kelvin's method of finding the potentials of attracting bodies by Inversion.

THEOR. II. *Let any body be changed into another body by inversion with regard to any point O. If the densities at corresponding points P, P′ be denoted by* ρ, ρ', *and their distances from O by* r, r', *let* $\rho'=\rho\left(\frac{\kappa}{r'}\right)^n$. *Then the moment of inertia of the second body with regard to any point C′ is equal to that of the first body with regard to the corresponding point C multiplied by either of the equal quantities* $\left(\frac{\kappa}{OC}\right)^2$ *or* $\frac{OC'}{OC}$. Here $n=8$, 6 or 4 according as the body is a *volume*, *area*, or *arc*.

To prove this, consider the case in which the body is a volume. By similar triangles $CP \,.\, r'=C'P' \,.\, OC$. We then find $\rho dv\,(CP)^2\left(\frac{\kappa}{OC}\right)^2=\rho' dv'\,(C'P')^2$, by proceeding as before. This being true for every element the theorem follows at once.

Ex. The density of a solid sphere varies inversely as the tenth power of the distance from an external point O. Prove that its moment of inertia about any straight line through O is the same as if the sphere were homogeneous and its density equal to that of the heterogeneous sphere at a point where the tangent from O meets the sphere. Prove that if the density had varied inversely as the sixth power of the distance from O, the masses of the two spheres would have been equal. What is the condition that they should have a common centre of gravity? [Math. Tripos.

47. **Centre of Pressure.** If a plane lamina is immersed in a homogeneous fluid it is proved in treatises on hydrostatics that the pressures on the elements of area act normally to the plane and are proportional to the product of the area of the element by the depth below a fixed horizontal plane often called "the effective surface." It easily follows from statical principles that the centre of these parallel forces lies in the plane of the lamina and is the same however the forces are turned round their points of application provided they remain parallel. *This point is called in hydrostatics the centre of pressure.*

Let the intersection of the lamina with the effective surface be taken as the axis of x and let the axis of y be in the plane of the lamina, the axes being rectangular. Then by the common formulæ for the centre of parallel forces

$$X=\frac{\text{Product of inertia about } Ox,\, Oy}{\text{moment of the area about } Ox},$$

$$Y=\frac{\text{Moment of inertia about } Ox}{\text{moment of the area about } Ox}.$$

Let the given area be equimomental to particles whose masses are m_1, m_2 &c. and let (x_1, y_1), (x_2, y_2) &c. be the coordinates of these particles. Then $$X = \frac{\Sigma mxy}{\Sigma my}, \quad Y = \frac{\Sigma my^2}{\Sigma my}.$$
But these are the formulæ to find the centre of gravity of particles whose masses are proportional to m_1y_1, m_2y_2 &c. having the same coordinates as before. Hence this rule,

If any area be equimomental to a series of particles, the centre of pressure of the area is the centre of gravity of the same particles with their masses increased in the ratio of their depths.

For example, *the centre of pressure of a triangle wholly immersed is the centre of gravity of three weights placed at the middle points of the sides and each proportional to the depth of the point at which it is placed.*

In this article we confine our attention to the hydrostatical properties of the point, but we may notice that the coordinates X and Y are so useful that in dynamics also names have been given to them. It follows from the formulæ (5) of the next article that X is the abscissa of the principal point of the axis of x, so that *the projection of the centre of pressure of any area on its intersection with the effective surface is the principal point of that intersection.* It will also be shown in Chap. III. that *the ordinate Y is equal to the distance of the centre of oscillation from the axis of suspension.* In this way we can translate our hydrostatical results into dynamics, and conversely.

Since the coordinates X, Y depend only on the ratio of the moments and products of inertia to the mass and on the position of the centre of gravity, it is clear that two equimomental areas have the same centre of pressure.

Ex. 1. If p, q, r be the depths of the corners of a triangular area wholly immersed in a fluid, prove that the areal coordinates of its centre of pressure referred to the sides of the triangle itself are $\frac{1}{4}(1+p/s)$, $\frac{1}{4}(1+q/s)$, $\frac{1}{4}(1+r/s)$, where $s=p+q+r$.

This may be proved by replacing the triangle by three weights situated at the middle points of the sides proportional to their depths, and taking moments about the sides in succession to find their centre of gravity.

Ex. 2. Let any vertical area be referred to Cartesian rectangular axes Ox, Oy, with the origin at the centre of gravity. Let the depth of the centre of gravity be h, and let the intersection of the area with the surface of the fluid make an angle θ with the axis of x, and let this intersection in the standard case cut the positive side of the axis of y. Let A, B and F be the moments and product of inertia of the area about the axes. Then by taking moments about Ox, Oy we see that the coordinates of the centre of pressure are

$$X = \frac{B \sin\theta - F\cos\theta}{h\alpha}, \quad Y = \frac{F\sin\theta - A\cos\theta}{h\alpha},$$

where α is the area.

Ex. 3. If the area turn round its centre of gravity in its own plane the locus of its centre of pressure *in the area* is an ellipse and *in space* is a circle. The ellipse has its principal diameters coincident in direction with the principal axes of the area at the centre of gravity. The circle has its centre in the vertical through the centre of gravity.

Ex. 4. In a heterogeneous fluid the pressure at any point P referred to a unit of area is given by $p=a+bz^n$ where z is the depth of P. Prove that the depth of the centre of pressure of any triangular area wholly immersed at any inclination to the horizon is $\dfrac{aH_1+bH_{n+1}}{aH_0+bH_n}$, where H_n is *the arithmetic mean* of the different homogeneous products of n dimensions of the depths z_1, z_2, z_3 of the three corners of the triangle.

Ex. 5. In rotating fluids the pressure at any point P is given by $p=a+bz+cr^2$, where r is the distance of P from the axis of z which is vertical. Show that the pressure on any part of the area of the containing vessel is given by

(1) whole pressure $=\int(a+bz+cr^2)\,d\sigma=(a+b\bar{z})\,\sigma+c\sigma k^2$,

where σ is the area of the part pressed, $\bar{z}$ the depth of its centre of gravity, and σk^2 the moment of inertia about the axis of z.

(2) Vertical pressure $=\iint(a+bz+cr^2)\,dx\,dy=aP+bV+cPk'^2$,

where P is the projection of σ on the plane of xy, V the volume between σ and its projection and Pk'^2 the moment of inertia of the projection P about the axis of z.

It is evident that in all these cases *the values of the integrals can in general be written down by the rules given in this chapter; so that actual integrations are for the most part unnecessary.*

48. **The Principal Axes of a system.** *A straight line being given it is required to find at what point in its length it is a principal axis of the system, and if any such point exist to find the other two principal axes at that point.* This point may be conveniently called *the principal point of the straight line.*

Take the straight line as axis of z, and any point O in it as origin. Let C be the point at which it is a principal axis, and let Cx', Cy' be the other two principal axes.

Let $CO=h$, $\theta=$ angle between Cx' and Ox. Then

$$\left.\begin{aligned} x' &= x\cos\theta+y\sin\theta \\ y' &= -x\sin\theta+y\cos\theta \\ z' &= z-h \end{aligned}\right\}.$$

Hence
$$\left.\begin{aligned}\Sigma mx'z' = \cos\theta\Sigma mxz+\sin\theta\Sigma myz \\ -h(\cos\theta\Sigma mx+\sin\theta\Sigma my)\end{aligned}\right\}=0\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned}\Sigma my'z' = -\sin\theta\Sigma mxz+\cos\theta\Sigma myz \\ -h(-\sin\theta\Sigma mx+\cos\theta\Sigma my)\end{aligned}\right\}=0\ldots\ldots\ldots\ldots(2),$$

$$\Sigma mx'y'=\Sigma m(y^2-x^2)\frac{\sin 2\theta}{2}+\Sigma mxy\cos 2\theta=0\ldots\ldots\ldots\ldots(3).$$

The last equation shows that

$$\tan 2\theta=\frac{2\Sigma mxy}{\Sigma m(x^2-y^2)}=\frac{2F}{B-A}\quad\ldots\ldots\ldots\ldots\ldots(4),$$

according to the previous notation.

The equations (1) and (2) must be satisfied by the same value of h. Eliminating h we get $\Sigma mxz \Sigma my = \Sigma myz \Sigma mx$ as the condition that the axis of z should be a principal axis at some point in its length. Substituting in (1) we have

$$h = \frac{\Sigma myz}{\Sigma my} = \frac{\Sigma mxz}{\Sigma mx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The equation (5) *expresses the condition that the axis of z should be a principal axis at some point in its length; and the value of h gives the position of this point.*

If $\Sigma mxz = 0$ and $\Sigma myz = 0$, the equations (1) and (2) are both satisfied by $h = 0$. *These are therefore the sufficient and necessary conditions that the axis of z should be a principal axis at the origin.*

If the system be a plane lamina and the axis of z be a normal to the plane at any point, we have $z = 0$. Hence the conditions $\Sigma mxz = 0$ and $\Sigma myz = 0$ are satisfied. Therefore one of the principal axes at any point of a plane lamina is a normal to the plane at that point.

In the case of a surface of revolution bounded by planes perpendicular to the axis, the axis is a principal axis at any point of its length.

Again, *equation* (4) *enables us, when one principal axis is given, to find the other two.* If $\theta = \alpha$ be the first value of θ, all the others are included in $\theta = \alpha + \frac{1}{2}n\pi$; hence all these values give only the same axes over again.

49. Since (4) does not contain h, it appears that if the axis of z be a principal axis at more than one point, the principal axes at those points are parallel. Again, in that case (5) must be satisfied by more than one value of h. But, since h enters only in the first power, this cannot be unless

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mxz = 0, \quad \Sigma myz = 0;$$

so that the axis must pass through the centre of gravity and be a principal axis at the origin, and therefore (since the origin is arbitrary) a principal axis at every point in its length.

If the principal axes at the centre of gravity be taken as the axes of x, y, z, (1) and (2) are satisfied for all values of h. Hence, if a straight line be a principal axis at the centre of gravity, it is a principal axis at every point in its length.

If the given straight line is parallel to a principal axis at the centre of gravity G, it is easy to see that the given line is a principal axis at the projection of G on itself. For let the origin O be taken at the projection, and let $G\xi$, $G\eta$, $G\zeta$ be a parallel system of axes, then since $\Sigma m\xi\zeta$, $\Sigma m\eta\zeta$ and $\bar{z}$ are zero, it follows from Art. 13 that Σmxz and Σmyz are also zero.

50. Let the system be projected on a plane perpendicular to the given straight line, so that the ratios of the elements of mass

to each other are unaltered. The given straight line, which has been taken as the axis of z, cuts this plane in O, and will be a principal axis of the projection at O, because, the projected system being a plane lamina, the conditions $\Sigma mxz = 0$, $\Sigma myz = 0$ are both satisfied. Since z does not appear in equation (4), it follows that, if the given straight line be a principal axis at some point C in its length, the other two principal axes at C will be parallel to the principal axes of the projected system at O. These last may often be conveniently found by the next proposition.

51. Ex. 1. The principal axes of a right-angled triangle at the right angle are, one perpendicular to the plane and two others inclined to its sides at the angles $\frac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}$, where a and b are the sides of the triangle adjacent to the right angle.

We have $\tan 2\theta = \frac{2F}{B-A}$, Art. 48, and by Art. 35, $A = M\frac{a^2}{6}$, $B = M\frac{b^2}{6}$, $F = M\frac{ab}{12}$.

Ex. 2. The principal axes of a quadrant of an ellipse at the centre are, one perpendicular to the plane and two others inclined to the principal diameters at the angles $\frac{1}{2}\tan^{-1}\frac{4}{\pi}\frac{ab}{a^2-b^2}$, where a and b are the semi-axes of the ellipse.

Ex. 3. The principal axes of a cube at any point P are, the straight line joining P to O the centre of gravity of the cube, and any two straight lines at P perpendicular to PO, and perpendicular to each other.

Ex. 4. Prove that the locus of a point P at which one of the principal axes is parallel to a given straight line is a rectangular hyperbola in the plane of which the centre of gravity of the body lies, and one of whose asymptotes is parallel to the given straight line. But if the given straight line be parallel to one of the principal axes at the centre of gravity, the locus of P is that principal axis or the perpendicular principal plane.

Take the origin at the centre of gravity, and one axis of coordinates parallel to the given straight line.

Ex. 5. The principal point of any side AB of a triangular area ABC bisects the distance between the middle point of that side and the foot of the perpendicular from the opposite corner on the side.

Ex. 6. An edge of a tetrahedron will be a principal axis at some point in its length only when it is perpendicular to the opposite edge. [Jullien.

Conversely, if this condition be satisfied, the edge will be a principal axis at a point C, such that $OC = \frac{2}{3}ON$, where N is the middle point of the edge and O is the foot of the perpendicular distance between it and the opposite edge.

Ex. 7. The axes Ox, Oy are so placed that the product of inertia F or Σmxy is zero. If A and B are the moments of inertia about these axes, prove that the product of inertia about two perpendicular axes Ox', Oy' in the plane xy is

$$F' = \tfrac{1}{2}(A-B)\sin 2\theta,$$

where θ is the angle xOx' measured in the positive direction from Ox.

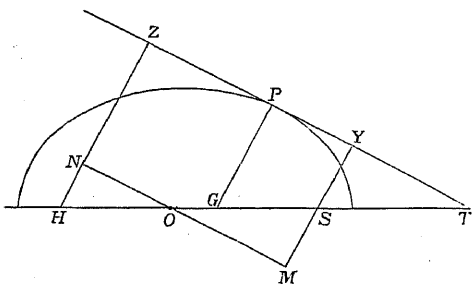
52. **Foci of Inertia.** *Given the positions of the principal axes Ox, Oy, Oz at the centre of gravity O, and the moments of inertia about them, to find the positions of the principal axes at any point P in the plane of xy, and the moments of inertia about them.*

Let the mass of the body be M, and let A, B be the moments of inertia about the axes Ox, Oy, of which we shall suppose A the greater. Let S, H be two points in the axis of greatest moment, one on each side of the origin so that $OS = OH = \sqrt{\frac{A-B}{M}}$. *These may be called the foci of inertia for that principal plane.*

Because these points are in one of the principal axes at the centre of gravity, the principal axes at S and H are parallel to the axes of coordinates, and the moments of inertia about those in the plane of xy are respectively A and $B + M \,.\, OS^2 = A$. These being equal, any straight line through S or H in the plane of xy is a principal axis at that point, and the moment of inertia about it is equal to A. See Arts. 16 and 23.

If P be any point in the plane of xy, then one of the principal axes at P will be perpendicular to the plane xy. For, if p, q be the coordinates of P, the conditions that this line should be a principal axis are $\Sigma m\,(x-p)\,z = 0, \quad \Sigma m\,(y-q)\,z = 0,$ which are obviously satisfied, because the centre of gravity is the origin, and the principal axes the axes, of coordinates.

The other two principal axes may be found thus. If two straight lines meeting at a point P be such that the moments of inertia about them are equal, then, provided they are in a principal plane, the principal axes at P bisect the angles between these two straight lines. For, if with centre P we describe the momental ellipse, the axes of this ellipse bisect the angles between any two equal radii vectores.



Join SP and HP; the moments of inertia about SP, HP are each equal to A. Hence, if PG and PT are the internal and external bisectors of the angle SPH, PG, PT are the principal axes at P. *If therefore with S and H as foci we describe any ellipse or hyperbola, the tangent and normal at any point are the principal axes at that point.*

53. Take any straight line MN through the origin, making an angle θ with the axis of x. Draw SM, HN perpendiculars on MN. The moment of inertia about MN is

$$= A\cos^2\theta + B\sin^2\theta = A - (A-B)\sin^2\theta$$
$$= A - M\,.\,(OS\sin\theta)^2 = A - M\,.\,SM^2.$$

Through P draw PT parallel to MN, and let SY and HZ be the perpendiculars from S and H on it. The moment of inertia about PT is then

$$\begin{aligned} &= \text{moment about } MN + M \,.\, MY^2 \\ &= A + M(MY - SM)(MY + SM) \\ &= A + M \,.\, SY \,.\, HZ. \end{aligned}$$

In the same way it may be proved that the moment of inertia about a line PG passing between H and S is *less* than A by the mass into the product of the perpendiculars from S and H on PG.

If therefore with S and H as foci we describe any ellipse or hyperbola, the moment of inertia about any tangent to either of these curves is constant.

It follows from this that the moments of inertia about the principal axes at P are equal to $B + \frac{1}{4}M(SP \pm HP)^2$.

For if a and b be the axes of the ellipse we have $a^2 - b^2 = OS^2 = (A - B)/M$, and hence

$$A + M \,.\, SY \,.\, HZ = A + Mb^2 = B + Ma^2 = B + \tfrac{1}{4}M(SP + HP)^2,$$

and the hyperbola may be treated in a similar manner.

54. This reasoning may be extended to points lying in any given plane passing through the centre of gravity O of the body. Let Ox, Oy be the axes in the given plane such that the product of inertia about them is zero (Art. 23). Construct the points S and H as before, so that OS^2 and OH^2 are each equal to the difference of the moments of inertia about Ox and Oy divided by the mass. Draw Sy' a parallel through S to the axis of y, the product of inertia about Sx, Sy' is equal to that about Ox, Oy together with the product of inertia of the whole mass collected at O. Both these are zero, hence the section of the momental ellipsoid at S is a circle, and the moment of inertia about every straight line through S in the plane xOy is the same and equal to that about Ox. We can then show that the moments of inertia about PH and PS are equal; so that PG, PT, the internal and external bisectors of the angle SPH, are the principal diameters of the section of the momental ellipsoid at P by the given plane. And it also follows that the moments of inertia about the tangents to a conic whose foci are S and H are the same.

55. Ex. 1. To find the foci of inertia of an elliptic area. The moments of inertia about the major and minor axes are $\frac{1}{4}Mb^2$ and $\frac{1}{4}Ma^2$. Hence the minor axis is the axis of greatest moment. The foci of inertia therefore lie in the *minor* axis at a distance from the centre $= \frac{1}{2}\sqrt{a^2 - b^2}$, i.e. half the distance of the geometrical foci from the centre.

Ex. 2. Two particles each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that the principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, provided that $\dfrac{m}{M} = \dfrac{5}{8}\,\dfrac{e^2}{1 - 2e^2}$.

Ex. 3. At the points which have been called foci of inertia *two* of the principal moments are equal. Show that it is not in general true that a point exists such that the moments of inertia about all axes through it are the same, and find the conditions that there may be such a point. *Such points when they exist in a solid body may be called the spherical points of inertia of that solid.*

Refer the body to the principal axes at the centre of gravity. Let P be the point required, (x, y, z) its coordinates. Since the momental ellipsoid at P is to be a sphere, the products of inertia about all rectangular axes meeting at P are zero. Hence, by Art. 13, $xy = 0$, $yz = 0$, $zx = 0$. It follows that two of the three x, y, z

must be zero, so that the point must be on one of the principal axes at the centre of gravity. Let this be called the axis of z. Since the moments of inertia about three axes at P parallel to the coordinate axes are $A+Mz^2$, $B+Mz^2$ and C, we see that these cannot be equal unless $A=B$ and each is less than C. There are then two points on the axis of unequal moment which are equimomental for all axes.

[Poisson and Binet.

Ex. 4. The spherical points of a hemispherical surface are the centre and a point on the surface. Find also the spherical points of a solid hemisphere.

By Art. 5, Ex. 8, the moments of inertia about every axis through the centre are the same. Hence the centre is one spherical point. Since the centre of gravity bisects the distance between the points the position of the other follows at once.

56. **Arrangement of Principal Axes.** *Given the positions of the principal axes at the centre of gravity O and the moments of inertia about them, to find the positions of the principal axes* and the principal moments at any other point P.*

Let the body be referred to its principal axes at the centre of gravity O, let A, B, C be its principal moments, the mass of the body being taken as unity. Construct a quadric confocal with the ellipsoid of gyration, and let the squares of its semi-axes be $a^2 = A+\lambda$, $b^2 = B+\lambda$, $c^2 = C+\lambda$. Let us find the moment of inertia with regard to any tangent plane.

Let (α, β, γ) be the direction angles of the perpendicular to any tangent plane. The moment of inertia, with regard to a parallel plane through O, is $\frac{1}{2}(A+B+C)-(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)$. The moment of inertia, with regard to the tangent plane, is found by adding the square of the perpendicular distance between the planes, viz. $(A+\lambda)\cos^2\alpha+(B+\lambda)\cos^2\beta+(C+\lambda)\cos^2\gamma$. We get

$$\left.\begin{array}{r}\text{moment of inertia with}\\ \text{regard to a tangent plane}\end{array}\right\} = \tfrac{1}{2}(A+B+C)+\lambda$$
$$= \tfrac{1}{2}(B+C-A)+a^2.$$

Thus the moments of inertia with regard to all tangent planes to any one quadric confocal with the ellipsoid of gyration are the same.

These planes are all principal planes at the point of contact. For draw any plane through the point of contact P, then in the case in which the confocal is an ellipsoid, the tangent plane parallel to this plane is more remote from the origin than this plane. Therefore, the moment of inertia with regard to any plane through P is less than the moment of inertia with regard to a tangent plane to the confocal ellipsoid through P. That is, the tangent plane to the ellipsoid is the principal plane of greatest moment. In the same way the tangent plane to the confocal

* Some of the following theorems were given by Lord Kelvin and Mr Townsend, in two articles which appeared at the same time in the *Mathematical Journal*, 1846. Their demonstrations are different from those given in this treatise. The theorem that the principal axes at P are normals to the three confocals is now ascribed in Thomson and Tait's *Treatise on Natural Philosophy* to Binet, *Journal de l'École Polytechnique*, 1811.

hyperboloid of two sheets through P is the principal plane of least moment. It follows that the tangent plane to the confocal hyperboloid of one sheet is the principal plane of mean moment.

Through a given point P, three confocals can be drawn, and the normals to these confocals are the principal axes at P. By Art. 5, Ex. 3, the principal axis of least moment is normal to the confocal ellipsoid and that of greatest moment normal to the confocal hyperboloid of two sheets.

57. The moment of inertia with regard to the *point* P is, by Art. 14, $\frac{1}{2}(A+B+C)+OP^2$. Hence, by Art. 5, Ex. 3, the moments of inertia about the normals to the three confocals through P whose parameters are $\lambda_1, \lambda_2, \lambda_3$ are respectively

$$OP^2-\lambda_1, \quad OP^2-\lambda_2, \quad OP^2-\lambda_3.$$

58. If we describe any other confocal and draw a tangent cone to it whose vertex is P, the axes of this cone are known to be the normals to three confocals through P. This gives another construction for the principal axes at P.

If the confocal diminish without limit, until it becomes a focal conic, we see that the principal axes of the system at P are the principal diameters of a cone whose vertex is P and base a focal conic of the ellipsoid of gyration at the centre of gravity.

Ex. Prove that the moment of inertia about any generator of the cone, vertex P, reciprocal to the tangent cone drawn from P to the ellipsoid of gyration is the same. [Math. Tripos, 1895.

59. If we wish to use only one quadric, we may consider the confocal ellipsoid through P. We know* that the normals to the other two confocals are tangents to the lines of curvature on the ellipsoid, and are also parallel to the principal diameters of the diametral section made by a plane parallel to the tangent plane at P. And if D_1, D_2 be these principal semi-diameters, we know that

$$\lambda_2=\lambda_1-D_1^2, \quad \lambda_3=\lambda_1-D_2^2.$$

Hence, if through any point P we describe the quadric

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda}+\frac{z^2}{C+\lambda}=1,$$

the axes of coordinates being the principal axes at the centre of gravity, then the principal axes at P are the normal to this quadric, and parallels to the axes of the diametral section made by a plane parallel to the tangent plane at P. And if these axes are $2D_1$ and $2D_2$, the principal moments at P are

$$OP^2-\lambda, \quad OP^2-\lambda+D_1^2, \quad OP^2-\lambda+D_2^2.$$

* A geometrical proof of the propositions required for this article was given in the former editions, but these results are now too well known to render this necessary.

Ex. If two bodies have the same centre of gravity, the same principal axes at the centre of gravity *and the differences of their principal moments equal*, each to each, then these bodies have the same principal axes at all points.

60. **Condition that a line should be a principal axis.** *The axes of coordinates being the principal axes at the centre of gravity it is required to express the condition that any given straight line may be a principal axis at some point in its length and to find that point.* Let the equations of the given straight line be

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then it must be a normal to some quadric

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda}+\frac{z^2}{C+\lambda}=1\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

at the point at which the straight line is a principal axis.

Hence comparing the equation of the normal to (2) with (1), we have

$$\frac{x}{A+\lambda}=\mu l,\quad \frac{y}{B+\lambda}=\mu m,\quad \frac{z}{C+\lambda}=\mu n\ \ldots\ldots\ldots(3).$$

These six equations must be satisfied by the same values of x, y, z, λ and μ. Substituting for x, y, z from (3) in (1), we get

$$A\mu-\frac{f}{l}=B\mu-\frac{g}{m}=C\mu-\frac{h}{n}.$$

Equating the values of μ given by these equations we have

$$\frac{\dfrac{f}{l}-\dfrac{g}{m}}{A-B}=\frac{\dfrac{g}{m}-\dfrac{h}{n}}{B-C}=\frac{\dfrac{h}{n}-\dfrac{f}{l}}{C-A}\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This clearly amounts to only one equation, and is the required condition that the straight line should be a principal axis at some point in its length.

Substituting for x, y, z from (3) in (2), we have

$$\lambda(l^2+m^2+n^2)=\frac{1}{\mu^2}-(Al^2+Bm^2+Cn^2),$$

which gives one value only to λ. The values of λ and μ having been found, equations (3) will determine x, y, z the coordinates of the point at which the straight line is a principal axis.

The geometrical meaning of this condition may be found by the following considerations, which were given by Townsend in the *Mathematical Journal*. The normal and tangent plane at every point of a quadric will meet any principal plane in a point and a straight line, which are pole and polar with regard to the focal conic in that plane. Hence, to find whether any assumed straight line is a principal axis or not, draw any plane perpendicular to the straight line and produce both the straight line and the plane to meet any principal plane at the centre of gravity. If the line of intersection of the plane be parallel to the polar

line of the point of intersection of the straight line with respect to the focal conic, the straight line will be a principal axis, if otherwise it will not be so. And the point at which it is a principal axis may be found by drawing a plane through the polar line perpendicular to the straight line. The point of intersection is the required point.

The analytical condition (4) exactly expresses the fact that the polar line is parallel to the intersection of the plane.

61. Ex. 1. Show that the straight line $a(x-a)=b(y-b)=c(z-c)$ is at some point in its length a principal axis of an ellipsoid whose semi-axes are a, b, c.

Ex. 2. Show that any straight line drawn on a lamina is a principal axis of that lamina at some point. Where is this point if the straight line pass through the centre of gravity?

Ex. 3. Given a plane $fx+gy+hz-1=0$, there is always some point in it at which it is a principal plane. Also this point is its intersection with the straight line $x/f-A=y/g-B=z/h-C$.

Ex. 4. Let two points P, Q be so situated that a principal axis at P intersects a principal axis at Q. Then if two planes be drawn at P and Q perpendicular to these principal axes, their intersection will be a principal axis at the point where it is cut by the plane containing the principal axes at P and Q. [Townsend.

For let the principal axes at P, Q meet any principal plane at the centre of gravity in p, q, and let the perpendicular planes cut the same principal plane in LN, MN. Also let the perpendicular planes intersect each other in RN. Then RN is perpendicular to the plane containing the points P, Q, p, q. Also since the polars of p and q are LN, MN, it follows that pq is the polar of the point N. Hence the straight line RN satisfies the criterion of the last article.

Ex. 5. If P be any point in a principal plane at the centre of gravity, then every axis which passes through P, and is a principal axis at some point, lies in one of two perpendicular planes. One of these planes is the principal plane at the centre of gravity, and the other is a plane perpendicular to the polar line of P with regard to the focal conic. Also the locus of all the points Q at which QP is a principal axis is a circle passing through P and having its centre in the principal plane. [Townsend.

Ex. 6. The edge of regression of the developable surface which is the envelope of the normal planes of any line of curvature drawn on a confocal quadric is a curve such that all its tangents are principal axes at some point in each.

62. **Locus of equal Moments.** *To find the locus of the points at which two principal moments of inertia are equal to each other.*

The principal moments at any point P are

$$I_1 = OP^2 - \lambda,\quad I_2 = OP^2 - \lambda + D_1^2,\quad I_3 = OP^2 - \lambda + D_2^2.$$

If we equate I_1 and I_2 we have $D_1 = 0$, and the point P must lie on the elliptic focal conic of the ellipsoid of gyration.

If we equate I_2 and I_3 we have $D_1 = D_2$, so that P is an umbilicus of any ellipsoid confocal with the ellipsoid of gyration. The locus of these umbilici is the hyperbolic focal conic.

In the first of these cases we have $\lambda = -C$, and D_2 is the semi-diameter of the focal conic conjugate to OP. Hence $D_2^2 + OP^2 =$ sum of squares of semi-axes $= A - C + B - C$. The three principal moments are therefore $I_1 = I_2 = OP^2 + C$, $I_3 = A + B - C$, and the axis of unequal moment is a tangent to the focal conic.

The second case may be treated in the same way by using a confocal hyperboloid, we therefore have $I_2 = I_3 = OP^2 + B$, $I_1 = A + C - B$, and the axis of unequal moment is a tangent to the focal conic.

These results follow also by combining Arts. 57 and 58. The cone which envelopes the ellipsoid of gyration and has its vertex at P must by these articles be a right cone if two principal moments at P are equal. But we know from solid geometry that this only happens when the vertex lies on a focal conic, and the unequal axis is then a tangent to that conic.

63. *To find the curves on any confocal quadric at which a principal moment of inertia is equal to a given quantity I.*

Firstly. The moment of inertia about a normal to a confocal quadric is $OP^2 - \lambda$. If this be constant, we have OP constant, and therefore the required curve is the intersection of that quadric with a concentric sphere. Such a curve is a sphero-conic.

Secondly. Let us consider those points at which the moment of inertia about a tangent is constant.

Construct any two confocals whose semi-major axes are a and a'. Draw any two tangent planes to these which cut each other at right angles. The moment of inertia about their intersection is the sum of the moments of inertia with regard to the two planes, and is therefore $B + C - A + a^2 + a'^2$. *Thus the moments of inertia about the intersections of perpendicular tangent planes to the same confocals are equal to each other.*

Let a, a', a'' be the semi-major axes of the three confocals which meet at any point P, then since confocals cut at right angles the moment of inertia about a tangent to the intersection of the confocals a', a'' is $I_1 = B + C - A + a'^2 + a''^2$.

The intersection of these two confocals is a line of curvature on either. *Hence the moments of inertia about the tangents to any line of curvature are equal to one another; and these tangents are principal axes at the point of contact.*

On the quadric a draw a tangent PT making angles ϕ and $\frac{1}{2}\pi - \phi$ with the tangents to the lines of curvature at the point of contact P. If I_2, I_3 be the moments about the tangents to these lines of curvature, the moment of inertia about the tangent $PT = I_2 \cos^2 \phi + I_3 \sin^2 \phi$

$$= B + C - A + (a''^2 + a^2)\cos^2 \phi + (a^2 + a'^2)\sin^2 \phi.$$

But, along a geodesic on the quadric a, $a'^2 \sin^2 \phi + a''^2 \cos^2 \phi$ is constant. *Hence the moments of inertia about the tangents to any geodesic on the quadric are equal to each other.*

64. Ex. 1. If a straight line touch any two confocals whose semi-major axes are a, a', the moment of inertia about it is $B + C - A + a^2 + a'^2$.

Ex. 2. When a body is referred to its principal axes at the centre of gravity, show how to find the coordinates of the point P at which the three principal moments are equal to the three given quantities I_1, I_2, I_3. [Jullien's Problem.

The elliptic coordinates of P are evidently $a^2=\frac{1}{2}(I_2+I_3-I_1-B-C+A)$, &c.; and the coordinates (x, y, z) may then be found by Salmon's formulæ,

$$x^2=\frac{a^2a'^2a''^2}{(A-B)(A-C)}, \text{ \&c.}$$

Ex. 3. Let two planes at right angles touch two confocals whose semi-major axes are a, a'; and let a, a' be the values of a, a' for confocals touching the intersection of the planes; then $a^2+a'^2=\mathrm{a}^2+\mathrm{a}'^2$, and the product of inertia with regard to the two planes is $(a^2a'^2-\mathrm{a}^2\mathrm{a}'^2)^{\frac{1}{2}}$.

65. **Equimomental Surface.** The locus of all those points at which one of the principal moments of inertia of the body is equal to a given quantity is called an *equimomental surface.*

To find the equation to such a surface we have only to put I_1 constant, this gives $\lambda=r^2-I$. Substituting in the equation of the confocal quadric, the equation of the surface becomes

$$\frac{x^2}{x^2+y^2+z^2+A-I}+\frac{y^2}{x^2+y^2+z^2+B-I}+\frac{z^2}{x^2+y^2+z^2+C-I}=1.$$

Through any point P on an equimomental surface describe a confocal quadric such that the principal axis is a tangent to a line of curvature on the quadric. By Art. 63, one of the intersections of the equimomental surface and this quadric is the line of curvature. Hence the principal axis at P about which the moment of inertia is I is a tangent to the equimomental surface.

Again, construct the confocal quadric through P such that the principal axis is a normal at P, then one of the intersections of the momental surface and this quadric is the sphero-conic through P. The normal to the quadric, being the principal axis, has just been shown to be a tangent to the surface. Hence the tangent plane to the equimomental surface is the plane which contains the normal to the quadric and the tangent to the sphero-conic.

To draw a perpendicular from the centre O on this tangent plane we may follow Euclid's rule. Take PP' a tangent to the sphero-conic, drop a perpendicular from O on PP', this is the radius vector OP, because PP' is a tangent to the sphere. At P in the tangent plane draw a perpendicular to PP', this is the normal PQ to the quadric. From O drop a perpendicular OQ on this normal, then OQ is a normal to the tangent plane. Hence this construction:

If P be any point on an equimomental surface whose parameter is I, and OQ a perpendicular from the centre on the tangent plane, then PQ is the principal axis at P about which the moment of inertia is I.

The equimomental surface becomes Fresnel's wave surface when I is greater than the greatest principal moment of inertia at the centre of gravity. The general form of the surface is too well known to need a minute discussion here. It consists of two sheets, which become a concentric sphere and a spheroid when two of the principal moments at the centre of gravity are equal. When the principal moments are unequal, there are two singularities in the surface.

(1) The two sheets meet at a point P in the plane of the greatest and least moments. At P there is a tangent cone to the surface. Draw any tangent plane to this cone, and let OQ be a perpendicular from the centre of gravity O on this tangent plane. Then PQ is a principal axis at P. Thus there are an infinite number of principal axes at P because an infinite number of tangent planes can be drawn to the cone. But at any given point there cannot be more than three principal axes unless two of the principal axes be equal, and then the locus of the

principal axes is a plane. Hence the point P is situated on a focal conic, and the locus of all the lines PQ is a normal plane to the conic. The point Q lies on a sphere whose diameter is OP, hence the locus of Q is a circle.

(2) The two sheets have a common tangent plane which touches the surface along a curve. This curve is a circle whose plane is perpendicular to the plane of greatest and least moments. Let OP' be a perpendicular from O on the plane of the circle, then P' is a point on the circle. If R be any other point on the circle the principal axis at R is RP'. Thus there is a circular ring of points, at each of which the principal axis passes through the same point, and the moments of inertia about these principal axes are all equal.

The equation to the equimomental surface may also be used for the purpose of finding the three principal moments at any point whose coordinates (x, y, z) are given. If we clear the equation of fractions, we have to determine I a cubic whose roots are the three principal moments.

Thus let it be required to find the locus of all those points at which any symmetrical function of the three principal moments is equal to a given quantity. We may express this symmetrical function in terms of the coefficients of the cubic by the usual rules, and the equation of the locus is found.

Ex. 1. If an equimomental surface cut a quadric confocal with the ellipsoid of gyration at the centre of gravity, then the intersections are a sphero-conic and a line of curvature. But, if the quadric be an ellipsoid, these cannot be both real.

For if the surface cut the ellipsoid in both, let P be a point on the line of curvature, and P' a point on the sphero-conic, then by Art. 59, $OP^2+D_1^2=OP'^2$, which is less than $A+\lambda$. But $OP^2+D_1^2+D_2^2=A+B+C+3\lambda$, therefore $D_2^2>B+C+2\lambda$, which is $>A+2\lambda$. Hence $D_2 >$ the greatest radius vector of the ellipsoid, which is impossible.

Ex. 2. Find the locus of all those points in a body at which

(1) the sum of the principal moments is equal to a given quantity I,

(2) the sum of the products of the principal moments taken two and two together is equal to I^2, (3) the product of the principal moments is equal to I^3.

The results are (1) by Art. 13, a sphere whose radius is $\{(I-A-B-C)/M\}^{\frac{1}{2}}$,

(2) by Art. 65, the surface

$$(x^2+y^2+z^2)^2+(A+B+C)(x^2+y^2+z^2)+Ax^2+By^2+Cz^2+AB+BC+CA=I^2,$$

(3) the surface $A'B'C'-A'y^2z^2-B'z^2x^2-C'x^2y^2-2x^2y^2z^2=I^3$,

where $A'=A+y^2+z^2$, with similar expressions for B', C'.

CHAPTER II

D'ALEMBERT'S PRINCIPLE, ETC.

66. THE principles, by which the motion of a single particle under the action of given forces can be determined, will be found discussed in any treatise on dynamics of a particle. These principles are called the three laws of motion. It is shown that if (x, y, z) be the coordinates of the particle at any time t referred to three rectangular axes fixed in space, m its mass, X, Y, Z the forces resolved parallel to the axes, the motion may be found by solving the simultaneous equations,

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z.$$

If we regard a rigid body as a collection of material particles connected by invariable relations, we may write down the equations of the several particles in accordance with the principles just stated. The forces on each particle are however no longer known, some of them being due to the mutual actions of the particles.

We assume (1) that the action between two particles is along the line which joins them, (2) that the action and reaction between any two are equal and opposite. Suppose there are n particles, then there will be $3n$ equations, and, as shown in any treatise on statics, $3n - 6$ unknown reactions. To find the motion it will be necessary to eliminate these unknown quantities. We shall thus obtain six resulting equations, and these will be shown, a little further on, to be sufficient to determine the motion of the body.

When there are several rigid bodies which mutually act and react on each other the problem becomes still more complicated. But it is unnecessary for us to consider in detail either this or the preceding case, for D'Alembert has proposed a method by which all the necessary equations may be obtained without writing down the equations of motion of the several particles, and without making any assumption as to the nature of the mutual actions except the following, which may be regarded as a natural consequence of the laws of motion:

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

67. *To explain D'Alembert's principle.*

In the application of this principle it will be convenient to use the term *effective force*, which may be defined as follows.

When a particle is moving as part of a rigid body, it is acted on by the external impressed forces and also by the molecular reactions of the other particles. If we consider this particle to be separated from the rest of the body, and all these forces removed, there is some one force which, under the same initial conditions, would make it move in the same way as before. This force is called the effective force on the particle. It is evidently the resultant of the impressed and molecular forces on the particle.

Let m be the mass of the particle, (x, y, z) its coordinates referred to any fixed rectangular axes at the time t. The accelerations of the particle are $\dfrac{d^2x}{dt^2}$, $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$. Let f be the resultant of these, then, as explained in dynamics of a particle, the effective force is measured by mf.

Let F be the resultant of the impressed forces, R the resultant of the molecular forces on the particle. Then mf is the resultant of F and R. Hence if mf be reversed, the three F, R and mf are in equilibrium.

We may apply the same reasoning to every particle of each body of the system. We thus have a group of forces similar to R, a group similar to F, and a group similar to mf, the three groups forming a system of forces in equilibrium. Now by D'Alembert's principle the group R will itself form a system of forces in equilibrium. Whence it follows that the group F will be in equilibrium with the group mf. Hence

If forces equal to the effective forces but acting in exactly opposite directions were applied at each point of the system these would be in equilibrium with the impressed forces.

By this principle the solution of a dynamical problem is reduced to that of a problem in statics. The process is as follows. We first choose some quantities by means of which the position of the system in space may be determined. We then express the effective forces on each element in terms of these quantities. These, when reversed, will be in equilibrium with the given impressed forces. Lastly, the equations of motion for each body may be formed, as is usually done in statics, by resolving in three directions and taking moments about three straight lines.

68. Before the publication of D'Alembert's principle a vast number of dynamical problems had been solved. These may be found scattered through the early volumes of the Memoirs of St Petersburg, Berlin and Paris, in the works of John Bernoulli and the *Opuscula* of Euler. They require for the most part the

determination of the motions of several bodies with or without weight which push or pull each other by means of threads or levers to which they are fastened or along which they can glide, and which having a certain impulse given them at first are then left to themselves or are compelled to move in given lines or surfaces.

The postulate of Huyghens, "that if any weights are put in motion by the force of gravity they cannot move so that the centre of gravity of them all shall rise higher than the place from which it descended," was generally one of the principles of the solution: but other principles were always needed in addition to this, and it required the exercise of ingenuity and skill to detect the most suitable in each case. Such problems were for some time a sort of trial of strength among mathematicians. The *Traité de dynamique* published by D'Alembert in 1743 put an end to this kind of challenge by supplying a direct and general method of resolving, or at least throwing into equations, any imaginable problem. The mechanical difficulties were in this way reduced to difficulties of pure mathematics. See Montucla, Vol. III. page 615, or Whewell's version in his *History of the Inductive Sciences*.

D'Alembert uses the following words:—"Soient A, B, C, &c. les corps qui composent le systême, et supposons qu'on leur ait imprimé les mouvemens, a, b, c, &c. qu'ils soient forcés, à cause de leur action mutuelle, de changer dans les mouvemens a, b, c, &c. Il est clair qu'on peut regarder le mouvement a imprimé au corps A comme composé du mouvement a, qu'il a pris, et d'un autre mouvement α; qu'on peut de même regarder les mouvemens b, c, &c. comme composés des mouvemens b, β; c, γ; &c., d'où il s'ensuit que le mouvement des corps A, B, C, &c. entr'eux auroit été le même, si au lieu de leur donner les impulsions a, b, c, on leur eût donné à-la-fois les doubles impulsions a, α; b, β; &c. Or par la supposition les corps A, B, C, &c. ont pris d'eux-mêmes les mouvemens a, b, c, &c. donc les mouvemens α, β, γ, &c. doivent être tels qu'ils ne dérangent rien dans les mouvemens a, b, c, &c. c'est-à-dire que si les corps n'avoient reçu que les mouvemens α, β, γ, &c. ces mouvemens auroient dû se détruire mutuellement, et le systême demeurer en repos. De là résulte le principe suivant pour trouver le mouvement de plusieurs corps qui agissent les uns sur les autres. Décomposez les mouvemens a, b, c, &c. imprimés à chaque corps, chacun en deux autres a, α; b, β; c, γ; &c. qui soient tels que si l'on n'eût imprimé aux corps que les mouvemens a, b, c, &c. ils eussent pu conserver les mouvemens sans se nuire réciproquement; et que si on ne leur eût imprimé que les mouvemens α, β, γ, &c. le systême fût demeuré en repos; il est clair que a, b, c, &c. seront les mouvemens que ces corps prendront en vertu de leur action. Ce qu'il falloit trouver."

69. The following remarks on D'Alembert's principle have been supplied by Sir G. Airy:

I have seen some statements of or remarks on this principle which appear to me to be erroneous. The principle itself is not a new physical principle, nor any addition to existing physical principles; but is a convenient principle of combination of mechanical considerations, which results in a comprehensive process of great elegance.

The tacit idea, which dominates through the investigation, is this:—That every mass of matter in any complex mechanical combination may be conceived as containing in itself two distinct properties:—one that of connexion in itself, of susceptibility to pressure-force, and of connexion with other such masses, but not of inertia nor of impressions of momentum:—the other that of discrete molecules of matter, held in their places by the connexion-frame, susceptible to externally impressed momentum, and possessing inertia. The union produces an imponderable skeleton, carrying ponderable particles of matter.